

A HOLOMORPHIC VERTEX OPERATOR ALGEBRA OF CENTRAL CHARGE 24 WITH WEIGHT ONE LIE ALGEBRA $F_{4,6}A_{2,2}$

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ABSTRACT. In this paper, holomorphic vertex operator algebra U of central charge 24 with weight one Lie algebra $A_{8,3}A_{2,1}^2$ is proved to be unique. Moreover, a holomorphic vertex operator algebra of central charge 24 with weight one Lie algebra $F_{4,6}A_{2,2}$ is obtained by applying a \mathbb{Z}_2 -orbifold construction to U . As a consequence, we verify that all 71 Lie algebras in Schellekens' list can be realized as the weight one Lie algebras of some holomorphic vertex operator algebras of central charge 24.

1. INTRODUCTION

The classification of strongly regular holomorphic vertex operator algebras (VOA) of central charge 24 is an important problem in the theory of vertex operator algebras. In 1993, Schellekens [54] obtained a partial classification and determined the possible Lie algebra structures for the weight one subspaces of holomorphic VOAs of central charge 24 (see also [24]). There are 71 Lie algebras in his list but only 39 of the 71 cases in his list have been constructed explicitly at that time. In the recent years, many new holomorphic VOAs of central charge 24 have been constructed. In [39, 40], a class of holomorphic VOAs called framed VOAs were studied. In particular, 17 holomorphic VOAs were constructed. In addition, holomorphic VOAs with weight one Lie algebras $E_{6,3}G_{2,1}^3$, $A_{2,3}^6$ and $A_{5,3}D_{4,3}A_{1,1}^3$ have been constructed in [50, 53] using \mathbb{Z}_3 -orbifold constructions associated with lattice VOAs. Recently, van Ekeren, Möller and Scheithauer [24] have established the general \mathbb{Z}_n -orbifold construction for elements of arbitrary orders and the constructions of holomorphic VOAs with the weight one Lie algebras $E_{6,4}C_{2,1}A_{2,1}$, $A_{4,5}^2$, $A_{2,6}D_{4,12}$, $A_{1,1}C_{5,3}G_{2,2}$ and $C_{4,10}$ were also discussed. In [41], the constructions of five other holomorphic VOAs have been obtained using an orbifold construction associated with inner automorphisms. Moreover, a holomorphic VOA of central charge 24 whose weight one Lie algebra has the type $A_{6,7}$ has been constructed in [42]. Based on these results, 70 of 71 cases in Schellekens' list have been

C. H. Lam was partially supported by MoST grant 104-2115-M-001-004-MY3 of Taiwan.

X. Lin is an "Overseas researchers under Postdoctoral Fellowship of Japan Society for the Promotion of Science", and is supported by JSPS Grant No. 16F16020.

constructed. There is only one remaining case and the corresponding Lie algebra has the type $F_{4,6}A_{2,2}$.

In this article, we shall construct a strongly regular holomorphic vertex operator algebra of central charge 24 whose weight one Lie algebra has the type $F_{4,6}A_{2,2}$. As a consequence, we verify that all 71 Lie algebras in Schellekens' list can be realized as the weight one Lie algebras of some strongly regular holomorphic vertex operator algebras of central charge 24. Our method is basically a \mathbb{Z}_2 -orbifold construction. We shall show that a strongly regular holomorphic VOA $\tilde{U}(g)$ of central charge 24 with $\tilde{U}(g)_1 = F_{4,6}A_{2,2}$ can be constructed by applying a \mathbb{Z}_2 -orbifold construction to a holomorphic VOA U with the weight one Lie algebra $A_{8,3}A_{2,1}^2$ and a suitable automorphism g of order 2. However, there are some fundamental differences between our method and the previous constructions for the other cases. In our construction, the fixed point of the automorphism g on $U_1 = A_{8,3}A_{2,1}^2$ should have the type $B_{4,6}A_{2,2}$. It means $g|_{U_1}$ is an outer automorphism of the Lie algebra U_1 . In general, it is very difficult to determine if an outer automorphism of Lie algebra U_1 can be lifted to an automorphism of the whole VOA. For $U_1 = A_{8,3}A_{2,1}^2$, there are at least two different constructions of a holomorphic VOA U with $U_1 = A_{8,3}A_{2,1}^2$. One construction is based on orbifold method and is obtained in [41]. Another construction is based on mirror extensions of VOAs [11, 57]. The construction based on mirror extensions is first obtained by Xu [57] in terms of conformal nets and is proposed in [11] in VOA setting. In this article, we shall use the construction based on mirror extensions of VOAs. Using mirror extensions and the theory of modular invariants, we shall show that the VOA structure of a holomorphic VOA U of central charge 24 with $U_1 = A_{8,3}A_{2,1}^2$ is unique, up to isomorphism (cf. Theorem 4.15). We also generalize a result of Shimakura [55, Proposition 3.2], which gives a sufficient condition for lifting of an automorphism of a subVOA to the whole VOA (cf. Theorem 3.7). By these two facts, we are able to show if U is a strongly regular holomorphic VOA of central charge 24 such that $U_1 = A_{8,3}A_{2,1}^2$, then there exists an involution $g \in \text{Aut}(U)$ such that U_1^g is a Lie algebra of type $B_{4,6}A_{2,2}$. Moreover, we are able to determine the conformal weights of the unique irreducible g -twisted $U^T(g)$ of U using the explicit action of g on U_1 .

The organization of this article is as follows. In Section 2, we recall some basic facts about vertex operator algebras. In Section 3, we study some properties of the mirror extension $\widetilde{L_{sl_9}(3,0)}$ of affine vertex operator algebra $L_{sl_9}(3,0)$. The vertex operator algebra structure of $\widetilde{L_{sl_9}(3,0)}$ is proved to be unique. We also show that the automorphism θ of $L_{sl_9}(3,0)$ can be lifted to an automorphism of $\widetilde{L_{sl_9}(3,0)}$. In Section 4, we study the structure of holomorphic vertex operator algebra U of central charge 24 with Lie algebra

$A_{8,3}A_{2,1}^2$. It is proved that holomorphic vertex operator algebra of central charge 24 with Lie algebra $A_{8,3}A_{2,1}^2$ is unique. As an application, we obtain an automorphism $\widetilde{\theta \otimes \sigma}$ of U . In Section 5, we determine the conformal weights of $\widetilde{\theta \otimes \sigma}$ -twisted irreducible U -modules. In Section 6, we construct a holomorphic vertex operator algebra of central charge 24 with Lie algebra $F_{4,6}A_{2,2}$ by orbifold construction.

2. PRELIMINARIES

2.1. Basic definitions. In this subsection, we shall recall some notations about vertex operator algebras from [25], [26], [44] and [58]. Let $(V, Y(\cdot, z), \mathbf{1}, \omega)$ be a vertex operator algebra as defined in [26]. The vacuum vector and the conformal element of V are denoted by $\mathbf{1}$ and ω , respectively. The vertex operator $Y(v, z)$ corresponding to $v \in V$ is expanded as $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$. We also use the standard notation $L(n)$ to denote the component operator of $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$. A linear automorphism σ of V is called an *automorphism* of V if $\sigma(\mathbf{1}) = \mathbf{1}$, $\sigma(\omega) = \omega$ and $\sigma(u_n v) = \sigma(u)_n \sigma(v)$ for any $u, v \in V$, $n \in \mathbb{Z}$. We denote the group of all automorphisms of V by $Aut(V)$.

For a vertex operator algebra V , a *weak V -module* is a vector space M equipped with a linear map

$$Y_M : V \rightarrow (\text{End} M)[[z, z^{-1}]],$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \quad v_n \in \text{End} M$$

satisfying a number of conditions (cf. [16], [25]). A weak V -module M is called an *admissible V -module* if M has a $\mathbb{Z}_{\geq 0}$ -gradation $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n)$ such that

$$a_m M(n) \subset M(\text{wt} a + n - m - 1)$$

for any homogeneous $a \in V$ and $m, n \in \mathbb{Z}$, where $\text{wt}(a) = s$ if $a \in V_s$. If any admissible V -module is a direct sum of irreducible admissible modules, then V is called *rational*. It was proved in [16] that if V is rational then there are only finitely many irreducible admissible V -modules up to isomorphism.

A *V -module* is a weak V -module M which carries a \mathbb{C} -grading induced by the spectrum of $L(0)$, that is, $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ where $M_\lambda = \{w \in M \mid L(0)w = \lambda w\}$. Moreover, one requires that M_λ is finite dimensional and for fixed $\lambda \in \mathbb{C}$, $M_{\lambda+n} = 0$ for sufficiently small integer n .

A rational vertex operator algebra is said to be *holomorphic* if it itself is the only irreducible module up to isomorphism. A vertex operator algebra V is said to be of *CFT-type* if $V_0 = \mathbb{C}\mathbf{1}$ (note that $V_n = 0$ for all $n < 0$ if $V_0 = \mathbb{C}\mathbf{1}$ [19]), and is said to be *C_2 -cofinite* if the subspace $C_2(V) = \langle u_{-2}v \mid u, v \in V \rangle$ has finite codimension in V . A

V -module $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ is said to be *self-dual* if M is isomorphic to M' , where M' denotes the V -module such that $M' = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda^*$ and the vertex operator $Y_{M'}$ is defined by the property

$$\langle Y_{M'}(a, z)u', v \rangle = \langle u', Y_M(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})v \rangle,$$

for $a \in V, u' \in M'$ and $v \in M$. It is obvious that a holomorphic vertex operator algebra is simple and self-dual. A vertex operator algebra is said to be *strongly regular* if it is simple, self-dual, rational, C_2 -cofinite and of CFT-type. Note that a strongly regular vertex operator algebra is self-dual [45].

Let V be a CFT-type vertex operator algebra. It is well-known that V_1 has a Lie algebra structure such that $[u, v] = u_0v$ for any $u, v \in V_1$ (cf. [6]). Moreover, it was proved in [20] that V_1 is a reductive Lie algebra if V is a strongly regular vertex operator algebra.

We now recall the notions of intertwining operator and fusion rules from [25]. Let M^1, M^2, M^3 be admissible V -modules. An *intertwining operator* \mathcal{Y} of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$ is a linear map

$$\begin{aligned} \mathcal{Y} : M^1 &\rightarrow \text{Hom}(M^2, M^3)\{z\}, \\ w^1 &\mapsto \mathcal{Y}(w^1, z) = \sum_{n \in \mathbb{C}} w_n^1 z^{-n-1} \end{aligned}$$

satisfying a number of conditions (cf. [25]). We use $\mathcal{I}_{M^1, M^2}^{M^3}$ to denote the vector space of intertwining operators of type $\begin{pmatrix} M^3 \\ M^1 \ M^2 \end{pmatrix}$. If V is a rational vertex operator algebra and $\{M^i | 0 \leq i \leq p\}$ is the set of irreducible admissible V -modules, we define the *fusion rules* to be the formal product rules

$$M^i \times M^j = \sum_{0 \leq k \leq p} N_{M^i, M^j}^{M^k} M^k,$$

where $N_{M^i, M^j}^{M^k}$ denotes the dimension of $\mathcal{I}_{M^i, M^j}^{M^k}$. The *fusion ring* of V is defined to be the ring with $\{M^i | 0 \leq i \leq p\}$ as a basis and with the fusion rules as the structural constants. Let V be a rational vertex operator algebra and M an irreducible admissible V -module. If for any irreducible admissible module M^2 , there exists an irreducible admissible module M^3 such that $M \times M^2 = M^3$, then M is called a *simple current* module of V .

2.2. Modular invariance of trace functions. We now recall the modular invariance property of vertex operator algebra from [58]. Let V be a rational vertex operator

algebra and let M^0, \dots, M^p be all the irreducible V -modules. Then $M^i, 0 \leq i \leq p$, has the form

$$M^i = \bigoplus_{n=0}^{\infty} M_{\lambda_i+n}^i,$$

with $M_{\lambda_i}^i \neq 0$ for some number λ_i , which is called the *conformal weight* of M^i . Let $\mathfrak{H} = \{\tau \in \mathbb{C} | \text{Im}\tau > 0\}$ be the upper half plane. The trace function associated to M^i is defined as follows: For any homogeneous element $v \in V$ and $\tau \in \mathfrak{H}$,

$$Z_{M^i}(v, \tau) := \text{tr}_{M^i} o(v) q^{L(0)-c/24} = q^{\lambda_i-c/24} \sum_{n \in \mathbb{Z}^+} \text{tr}_{M_{\lambda_i+n}^i} o(v) q^n,$$

where $o(v) = v_{\text{wt}v-1}$ and $q = e^{2\pi\sqrt{-1}\tau}$. Assume further that V is a C_2 -cofinite vertex operator algebra, then $Z_{M^i}(v, \tau)$ converges to a holomorphic function on the domain $|q| < 1$ [17, 58].

Recall that the full modular group $SL(2, \mathbb{Z})$ has generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and acts on \mathfrak{H} as follows:

$$\gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Then the full modular group has an action on the trace functions. More precisely, we have the following result which was proved in [58] (also see [17]).

Theorem 2.1. *Let V be a rational and C_2 -cofinite vertex operator algebra with the irreducible V -modules M^0, \dots, M^p . Then the vector space spanned by $Z_{M^0}(v, \tau), \dots, Z_{M^p}(v, \tau)$ is invariant under the action of $SL(2, \mathbb{Z})$ defined above, i.e., there is a representation ρ of $SL(2, \mathbb{Z})$ on this vector space and the transformation matrices are independent of the choice of $v \in V$.*

2.3. Quantum dimensions. In this subsection, we recall some facts about quantum dimensions of irreducible modules of vertex operator algebras from [12]. Let V be a strongly regular vertex operator algebra and let $M^0 = V, M^1, \dots, M^p$ be all the inequivalent irreducible V -modules. The *quantum dimension* of M^i is defined to be

$$\text{qdim}_V M^i = \lim_{y \rightarrow 0^+} \frac{Z_{M^i}(\sqrt{-1}y)}{Z_V(\sqrt{-1}y)},$$

where y is real and positive. The *global dimension* of the vertex operator algebra V is defined to be

$$\text{Glob } V = \sum_{i=0}^p (\text{qdim}_V M^i)^2.$$

The following result was proved in [12].

Theorem 2.2. *Let V be a strongly regular vertex operator algebra and let $M^0 = V, M^1, \dots, M^p$ be all the irreducible V -modules. Assume further that the conformal weights of M^1, \dots, M^p are greater than 0. Then*

- (1) $\text{qdim } M^i \geq 1$ for any $0 \leq i \leq p$.
- (2) M^i is a simple current V -module if and only if $\text{qdim } M^i = 1$.

Recall that a vertex operator algebra U is called an *extension vertex operator algebra* of V if V is a vertex operator subalgebra of U and V, U have the same conformal element. Then we have the following result which was proved in [2].

Theorem 2.3. *Let U, V be strongly regular vertex operator algebras and let $M^0 = V, M^1, \dots, M^p$ be all the irreducible V -modules. Assume further that U is an extension vertex operator algebra of V and that the conformal weights of M^1, \dots, M^p are greater than 0. Then, we have*

$$\text{Glob}(U) = (\text{qdim}_V U)^2 \text{Glob}(V).$$

2.4. Modular invariants of vertex operator algebras. In this subsection, we recall some facts about modular invariants of vertex operator algebras from [18]. We shall assume that V is a strongly regular vertex operator algebra. Let $M^0 = V, M^1, \dots, M^p$ be all the irreducible V -modules. A *modular invariant* of V is a $(p+1) \times (p+1)$ -matrix X satisfying the following conditions:

- (M1) The entries of X are nonnegative integers.
- (M2) $X_{0,0} = 1$.
- (M3) $XS = SX$ and $XT = TX$, where we use S, T to denote the modular transformation matrices $\rho(S)$ and $\rho(T)$ respectively.

In the following, we shall construct a modular invariant of V from an extension vertex operator algebra of V . First, we have the following result (cf. [1], [32]).

Theorem 2.4. *Let V be a strongly regular vertex operator algebra and U be an extension vertex operator algebra of V . Assume further that U is simple. Then, U is rational and C_2 -cofinite.*

We now assume that U, V are strongly regular vertex operator algebras and that U is an extension vertex operator algebra of V . For $u, v \in V$, set

$$f_V(u, v, \tau_1, \tau_2) = \sum_{i=0}^p Z_{M^i}(u, \tau_1) \overline{Z_{M^i}(v, \tau_2)},$$

where $\tau_1, \tau_2 \in \mathfrak{H}$.

Similarly, for $u, v \in U$, set

$$f_U(u, v, \tau_1, \tau_2) = \sum_M Z_M(u, \tau_1) \overline{Z_M(v, \tau_2)},$$

where M ranges through the equivalent classes of irreducible U -modules. Since each irreducible U -module M is a direct sum of irreducible V -modules, there exists a matrix $X = (X_{i,j})$ such that $X_{i,j} \geq 0$ for all i, j and for $u, v \in V$,

$$f_U(u, v, \tau_1, \tau_2) = \sum_{i,j=0}^p X_{i,j} Z_{M^i}(u, \tau_1) \overline{Z_{M^j}(v, \tau_2)}.$$

Moreover, the matrix $X = (X_{i,j})$ is uniquely determined by the following proposition.

Proposition 2.5 ([18]). *Let M^0, \dots, M^p be all the V -irreducible modules. Set*

$$\mathbf{Z}(u, \tau) = (Z_{M^0}(u, \tau), \dots, Z_{M^p}(u, \tau))^T,$$

If $A = (a_{ij})$ is a matrix such that for any $u, v \in V$,

$$\mathbf{Z}(u, \tau_1)^T A \overline{\mathbf{Z}(v, \tau_2)} = 0,$$

then $A = 0$.

It was further shown in [18] that

Theorem 2.6. *The matrix X is a modular invariant of V .*

Furthermore, by the discussion above, we have the following simple observation.

Lemma 2.7. *Let V, U, X be as above. Suppose that there exist irreducible V -modules M^i and M^j such that $X_{i,j} \neq 0$. Then we have $X_{i,i} \neq 0$ and $X_{j,j} \neq 0$.*

2.5. Mirror extensions of vertex operator algebras. In this subsection, we shall recall from [48] some facts about mirror extensions of vertex operator algebras. First, we have the following result which was proved in [31].

Theorem 2.8. *Let V be a strongly regular vertex operator algebra and let \mathcal{C}_V be the category of V -modules. Then \mathcal{C}_V is a modular tensor category such that V is the unit object.*

Recall from [4] that for any modular tensor category (\mathcal{C}, \otimes) and an object M in \mathcal{C} , there is a natural isomorphism

$$\theta_M : M \longrightarrow M$$

such that

$$\begin{aligned}\theta_{M \otimes N} &= c_{N,M} c_{M,N} (\theta_M \otimes \theta_N), \\ \theta_1 &= \text{id}, \\ \theta_{M^*} &= (\theta_M)^*,\end{aligned}$$

where 1 denotes the unit object of \mathcal{C} , M^* denotes the dual of M , $c_{-, -}$ denotes the braiding of \mathcal{C} , and $(\theta_M)^* \in \text{Hom}(M^*, M^*)$ denotes the image of $\theta_M \in \text{Hom}(M, M)$ under the canonical map. It was shown in [32] that an extension vertex operator algebra U of V induces an etale algebra A_U (cf. [10]) in \mathcal{C}_V such that $\theta_{A_U} = \text{id}$. Moreover, we have the following result which was proved in [32] (see also [48]).

Theorem 2.9. *Let V be a strongly regular vertex operator algebra and let \mathcal{C}_V be the category of V -modules. Then the following statements are equivalent*

- (1) *There exists an extension vertex operator algebra U of V such that U viewed as a U -module is irreducible.*
- (2) *There exists an etale algebra A_U in \mathcal{C}_V such that A_U viewed as a V -module is isomorphic to U and that $\theta_{A_U} = \text{id}$.*

We now let $(U, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and $(V, Y, \mathbf{1}, \omega')$ be a vertex operator subalgebra of U such that $\omega' \in U_2$ and $L(1)\omega' = 0$. It was proved in [27] that $(V^c, Y, \mathbf{1}, \omega - \omega')$ is also a vertex operator subalgebra of U , where $V^c = C_U(V) = \{v \in U \mid \omega'_0 v = 0\}$. Assume further that $(V, Y, \mathbf{1}, \omega)$, $(V^c, Y, \mathbf{1}, \omega - \omega')$, $(U, Y, \mathbf{1}, \omega')$ are strongly regular and $(V^c)^c = V$, and denote the tensor products of the module categories \mathcal{C}_V , \mathcal{C}_{V^c} and $\mathcal{C}_{V \otimes V^c}$ by \boxtimes_V , \boxtimes_{V^c} , $\boxtimes_{V \otimes V^c}$, respectively. Then we have the following results which were proved in [48].

Theorem 2.10. (1) *As a $V \otimes V^c$ -module, U has the following decomposition*

$$U = V \otimes V^c \oplus (\oplus_{i=1}^n M^i \otimes N^i),$$

where $M^0 = V, M^1, \dots, M^n$ (resp. $N^0 = V^c, N^1, \dots, N^n$) are mutually inequivalent irreducible V -modules (resp. V^c -modules).

(2) *Let $\mathcal{K}(\mathcal{C}_V)$ and $\mathcal{K}(\mathcal{C}_{V^c})$ be the Grothendieck rings of \mathcal{C}_V and \mathcal{C}_{V^c} , respectively. Then $\mathbb{Z}M^0 \oplus \dots \oplus \mathbb{Z}M^n$ (resp. $\mathbb{Z}N^0 \oplus \dots \oplus \mathbb{Z}N^n$) forms a subring of $\mathcal{K}(\mathcal{C}_V)$ (resp. $\mathcal{K}(\mathcal{C}_{V^c})$).*

(3) *For any $0 \leq i_1, i_2, i_3 \leq n$, $N_{M^{i_1}, M^{i_2}}^{M^{i_3}} = N_{N^{i_1}, N^{i_2}}^{N^{i_3}}$.*

Let \mathcal{C}_V^0 (resp. $\mathcal{C}_{V^c}^0$) be the tensor subcategory of \mathcal{C}_V (resp. \mathcal{C}_{V^c}) such that the Grothendieck ring of \mathcal{C}_V^0 (resp. $\mathcal{C}_{V^c}^0$) is isomorphic to the subring $\mathbb{Z}M^0 \oplus \dots \oplus \mathbb{Z}M^n$ (resp. $\mathbb{Z}N^0 \oplus \dots \oplus \mathbb{Z}N^n$) of $\mathcal{K}(\mathcal{C}_V)$ (resp. $\mathcal{K}(\mathcal{C}_{V^c})$). Then we have the following results which were proved in [48].

Theorem 2.11. (1) *There is a braid-reversing equivalence $\mathcal{T} : \mathcal{C}_V^0 \rightarrow \mathcal{C}_{V^c}^0$ such that $\mathcal{T}(M^i) \cong (N^i)'$.*

(2) *If there is a vertex operator algebra structure $Y_{V^e}(\cdot, z)$ on the V -module*

$$V^e = V \oplus (\oplus_{i=1}^n m_i M^i),$$

where m_i 's are nonnegative integers, such that $(V^e, Y_{V^e}(\cdot, z))$ is an extension vertex operator algebra of V , then there exists a vertex operator algebra structure $Y_{(V^c)^e}(\cdot, z)$ on the V^c -module

$$(V^c)^e = V^c \oplus (\oplus_{i=1}^n m_i (N^i)'),$$

such that $((V^c)^e, Y_{(V^c)^e}(\cdot, z))$ is an extension vertex operator algebra of V^c . Moreover, $(V^c)^e$ is a simple vertex operator algebra if V^e is a simple vertex operator algebra.

Furthermore, we have the following result about the uniqueness.

Theorem 2.12. *Let V^e and $(V^c)^e$ be as above. Suppose that for any two extension vertex operator algebras $(V^e, Y_{V^e}^1(\cdot, z))$, $(V^e, Y_{V^e}^2(\cdot, z))$ of V , there exists a linear isomorphism $\phi : V^e \rightarrow V^e$ satisfying the following conditions*

$$\phi|_V = \text{id}, \quad \phi(Y_{V^e}^1(u^1, z)u^2) = Y_{V^e}^2(\phi(u^1), z)\phi(u^2), \quad \text{for any } u^1, u^2 \in V^e.$$

Then for any two extension vertex operator algebras $((V^c)^e, Y_{(V^c)^e}^1(\cdot, z))$, $((V^c)^e, Y_{(V^c)^e}^2(\cdot, z))$ of V^c , there exists a linear isomorphism $\phi^c : (V^c)^e \rightarrow (V^c)^e$ satisfying the following conditions

$$\phi^c|_{V^c} = \text{id}, \quad \phi^c(Y_{(V^c)^e}^1(v^1, z)v^2) = Y_{(V^c)^e}^2(\phi^c(v^1), z)\phi^c(v^2), \quad \text{for any } v^1, v^2 \in (V^c)^e.$$

Proof: Assume that there exist two vertex operator algebras $((V^c)^e, Y_{(V^c)^e}^1(\cdot, z))$ and $((V^c)^e, Y_{(V^c)^e}^2(\cdot, z))$ such that $((V^c)^e, Y_{(V^c)^e}^1(\cdot, z))$, $((V^c)^e, Y_{(V^c)^e}^2(\cdot, z))$ are extension vertex operator algebras of V^c . Let $A_{(V^c)^e}^1, A_{(V^c)^e}^2$ be the etale algebras in $\mathcal{C}_{V^c}^0$ induced from $((V^c)^e, Y_{(V^c)^e}^1(\cdot, z))$, $((V^c)^e, Y_{(V^c)^e}^2(\cdot, z))$, respectively. Since $\mathcal{T} : \mathcal{C}_V^0 \rightarrow \mathcal{C}_{V^c}^0$ is a braid-reversing equivalence, there exists a braid-reversing functor $\mathcal{G} : \mathcal{C}_{V^c}^0 \rightarrow \mathcal{C}_V^0$ such that $\mathcal{T} \circ \mathcal{G}$, $\mathcal{G} \circ \mathcal{T}$ are natural isomorphic to $\text{id}_{\mathcal{C}_{V^c}^0}$, $\text{id}_{\mathcal{C}_V^0}$, respectively, that is, there exist a family of isomorphisms $\eta^1(N) : \mathcal{T} \circ \mathcal{G}(N) \rightarrow N$, $N \in \mathcal{C}_{V^c}^0$, and $\eta^2(M) : \mathcal{G} \circ \mathcal{T}(M) \rightarrow M$, $M \in \mathcal{C}_V^0$ satisfying

$$\eta^1(N^2) \circ (\mathcal{T} \circ \mathcal{G}(g)) = g \circ \eta^1(N^1), \quad \eta^2(M^2) \circ (\mathcal{G} \circ \mathcal{T}(f)) = f \circ \eta^2(M^1),$$

for any $M^1, M^2 \in \mathcal{C}_V^0$, $N^1, N^2 \in \mathcal{C}_{V^c}^0$ and $f : M^1 \rightarrow M^2$, $g : N^1 \rightarrow N^2$ (see [37]). Note that, by Theorem 2.11, $\mathcal{G}(V^c)$ is isomorphic to V and that $\mathcal{G}(A_{(V^c)^e}^1)$, $\mathcal{G}(A_{(V^c)^e}^2)$ are two etale algebras in \mathcal{C}_V^0 such that $\mathcal{G}(A_{(V^c)^e}^1)$, $\mathcal{G}(A_{(V^c)^e}^2)$ viewed as V -modules are isomorphic to V^e . It follows from Theorem 2.9 that there exist two vertex operator

algebra structures $(\mathcal{G}((V^c)^e), Y^1(\cdot, z))$, $(\mathcal{G}((V^c)^e), Y^2(\cdot, z))$ such that $(\mathcal{G}((V^c)^e), Y^1(\cdot, z))$, $(\mathcal{G}((V^c)^e), Y^2(\cdot, z))$ are extension vertex operator algebras of $\mathcal{G}(V^c)$. By assumption, there exists a linear isomorphism $\phi : \mathcal{G}((V^c)^e) \rightarrow \mathcal{G}((V^c)^e)$ satisfying the following conditions

$$\phi|_{\mathcal{G}(V^c)} = \text{id}, \quad \phi(Y^1(u^1, z)u^2) = Y^2(\phi(u^1), z)\phi(u^2), \quad \text{for any } u^1, u^2 \in \mathcal{G}((V^c)^e).$$

Then we know that ϕ induces an etale algebra isomorphism $\tilde{\phi} : \mathcal{G}(A_{(V^c)^e}^1) \rightarrow \mathcal{G}(A_{(V^c)^e}^2)$ such that $\tilde{\phi}|_{\mathcal{G}(V^c)} = \text{id}$. Hence, $\mathcal{T}(\tilde{\phi})$ is an etale algebra isomorphism from $\mathcal{T} \circ \mathcal{G}(A_{(V^c)^e}^1)$ to $\mathcal{T} \circ \mathcal{G}(A_{(V^c)^e}^2)$ such that $\mathcal{T}(\tilde{\phi})|_{\mathcal{T}(\mathcal{G}(V^c))} = \text{id}$. Note that $\eta^1(A_{(V^c)^e}^1), \eta^1(A_{(V^c)^e}^2)$ are algebra isomorphism (see Definition XI.4.1 of [37]). As a result, $\eta^1(A_{(V^c)^e}^2) \circ \mathcal{T}(\tilde{\phi}) \circ \eta^1(A_{(V^c)^e}^1)^{-1}$ is an etale algebra isomorphism from $A_{(V^c)^e}^1$ to $A_{(V^c)^e}^2$ such that $\eta^1(A_{(V^c)^e}^2) \circ \mathcal{T}(\tilde{\phi}) \circ \eta^1(A_{(V^c)^e}^1)^{-1}|_{V^c} = \text{id}$. Therefore, $\eta^1(A_{(V^c)^e}^2) \circ \mathcal{T}(\tilde{\phi}) \circ \eta^1(A_{(V^c)^e}^1)^{-1}$ will induce the desired isomorphism. The proof is complete. \square

As a corollary, we have the following:

Corollary 2.13. *Let V^e and $(V^c)^e$ be as above. Suppose that there is a unique vertex operator algebra structure on V^e as an extension vertex operator algebra of V . Then there is a unique vertex operator algebra structure on $(V^c)^e$ as an extension vertex operator algebra of V^c .*

3. MIRROR EXTENSIONS OF AFFINE VERTEX OPERATOR ALGEBRA $L_{sl_9}(3\check{\Lambda}_0)$

3.1. Affine vertex operator algebras. In this subsection, we shall recall some facts about affine vertex operator algebras from [27] and [44]. Let \mathfrak{g} be a finite dimensional simple Lie algebra and $\langle \cdot, \cdot \rangle$ the normalized Killing form of \mathfrak{g} , i.e., $\langle \theta, \theta \rangle = 2$ for the highest root θ of \mathfrak{g} . Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} and denote the corresponding root system by $\Delta_{\mathfrak{g}}$ and the root lattice by Q . We further fix simple roots $\{\alpha_1, \dots, \alpha_l\}$, and denote the set of positive roots by $\Delta_{\mathfrak{g}}^+$. Then the weight lattice P of \mathfrak{g} is the set of $\lambda \in \mathfrak{h}$ such that $\frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$ for all $\alpha \in \Delta_{\mathfrak{g}}$. Note that P is equal to $\bigoplus_{i=1}^l \mathbb{Z}\Lambda_i$, where Λ_i are the fundamental weights defined by the equation $\frac{2\langle \Lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j}$. We also use the standard notation P_+ to denote the set of dominant weights $\{\Lambda \in P \mid \frac{2\langle \Lambda, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \geq 0, 1 \leq j \leq l\}$. For any $\alpha \in \Delta_{\mathfrak{g}}^+$, we fix $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ such that $[x_{\alpha}, x_{-\alpha}] = h_{\alpha}$, $[h_{\alpha}, x_{\pm\alpha}] = \pm 2x_{\pm\alpha}$, where $h_{\alpha} = \frac{2}{\langle \alpha, \alpha \rangle}\alpha$.

Recall that the affine Lie algebra associated to \mathfrak{g} is defined on $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{-1}, t] \oplus \mathbb{C}K$ with Lie brackets

$$[x(m), y(n)] = [x, y](m+n) + \langle x, y \rangle m \delta_{m+n, 0} K,$$

$$[K, \widehat{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$, where $x(n)$ denotes $x \otimes t^n$. In particular, $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t^{-1}, t] \oplus \mathbb{C}K$ is a subalgebra of $\widehat{\mathfrak{g}}$.

For a positive integer k and a weight $\Lambda \in P$, let $L_{\mathfrak{g}}(\Lambda)$ be the irreducible highest weight module for \mathfrak{g} with highest weight Λ and define

$$V_{\mathfrak{g}}(k, \Lambda) = \text{Ind}_{\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} L_{\mathfrak{g}}(\Lambda),$$

where $L_{\mathfrak{g}}(\Lambda)$ is viewed as a module for $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ such that $\mathfrak{g} \otimes t\mathbb{C}[t]$ acts as 0 and K acts as k . It is well-known that $V_{\mathfrak{g}}(k, \Lambda)$ has a unique maximal proper submodule which is denoted by $J(k, \Lambda)$ (see [35]). Let $L_{\mathfrak{g}}(k, \Lambda)$ be the corresponding irreducible quotient module. It was proved in [27] that $L_{\mathfrak{g}}(k, 0)$ has a vertex operator algebra structure such that the conformal element

$$\omega = \frac{1}{2(k + h^{\vee})} \left(\sum_{i=1}^{\dim \mathfrak{h}} u_i(-1)u_i(-1)\mathbf{1} + \sum_{\alpha \in \Delta_{\mathfrak{g}}} \frac{\langle \alpha, \alpha \rangle}{2} x_{\alpha}(-1)x_{-\alpha}(-1)\mathbf{1} \right),$$

where h^{\vee} denotes the dual Coxeter number of \mathfrak{g} and $\{u_i | 1 \leq i \leq \dim \mathfrak{h}\}$ is an orthonormal basis of \mathfrak{h} with respect to $\langle \cdot, \cdot \rangle$.

Theorem 3.1 ([27, 35]). *Let k be a positive integer. Then*

- (1) $L_{\mathfrak{g}}(k, 0)$ is a strongly regular vertex operator algebra;
- (2) $L_{\mathfrak{g}}(k, \Lambda)$ is a module for the vertex operator algebra $L_{\mathfrak{g}}(k, 0)$ if and only if $\Lambda \in P_+^k$, where $P_+^k = \{\Lambda \in P_+ | \langle \Lambda, \theta \rangle \leq k\}$;
- (3) If $L_{\mathfrak{g}}(k, \Lambda)$ is an $L_{\mathfrak{g}}(k, 0)$ -module such that $L_{\mathfrak{g}}(k, \Lambda) \not\cong L_{\mathfrak{g}}(k, 0)$, then the conformal weight of $L_{\mathfrak{g}}(k, \Lambda)$ is positive.

We next recall some facts about simple current modules of $L_{\mathfrak{g}}(k, 0)$. Let $\theta = \sum_{i=1}^l a_i \alpha_i$, $a_i \in \mathbb{Z}_+$, be the highest root. It is well-known that the irreducible $L_{\mathfrak{g}}(k, 0)$ -module $L_{\mathfrak{g}}(k, k\Lambda_i)$ is a simple current $L_{\mathfrak{g}}(k, 0)$ -module if $a_i = 1$ (see [15, 28, 29, 47]). In particular, for $\mathfrak{g} = sl_{n+1}$, $L_{\mathfrak{g}}(k, k\Lambda_1), \dots, L_{\mathfrak{g}}(k, k\Lambda_n)$ are simple current $L_{sl_{n+1}}(k, 0)$ -modules. Moreover, these are all the simple current $L_{sl_{n+1}}(k, 0)$ -modules (see [15], [47]).

3.2. Mirror extensions of affine vertex operator algebra $L_{sl_9}(3, 0)$. In this subsection, we shall construct some extension vertex operator algebra of $L_{sl_9}(3, 0)$. Consider the affine vertex operator algebra $L_{sl_{27}}(1, 0)$. It is well-known that $L_{sl_{27}}(1, 0)$ contains a vertex operator subalgebra isomorphic to $L_{sl_3}(9, 0) \otimes L_{sl_9}(3, 0)$. To determine the decomposition of $L_{sl_{27}}(1, 0)$ viewed as an $L_{sl_3}(9, 0) \otimes L_{sl_9}(3, 0)$ -module, we need to recall some notations from [52]. Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0 = \lambda_{k+1} = \dots)$ be a partition of $|\lambda| = \lambda_1 + \dots + \lambda_k$. We define $h(\lambda) = k$ and identify λ with its corresponding Young diagram; thus $h(\lambda)$ is just the number of rows in this diagram. We write I_n for the set of all partitions with $h(\lambda) \leq n$. Let $I_{n,m}$ be the set of all $\lambda \in I_n$ with $\lambda_1 \leq m$. Hence

$\lambda \in I_{n,m}$ if and only if its Young diagram fits into an $m \times n$ rectangle. Denote by λ^t the transposed partition of λ . Clearly, $\lambda \in I_{n,m}$ implies $\lambda^t \in I_{m,n}$.

Let $C_{n,m} = \{(a_0, a_1, \dots, a_{n-1}) \in \mathbb{N}^n \mid a_0 + \dots + a_{n-1} = m\}$. By identifying $a = (a_0, a_1, \dots, a_{n-1})$ with $\widehat{a} = a_1\Lambda_1 + \dots + a_{n-1}\Lambda_{n-1}$, we know that $C_{n,m}$ is exactly the set of dominant \widehat{sl}_n -weights of level m . For any $\lambda \in I_{n,m}$, we define

$$w_{n,m}(\lambda) = (m - \lambda_1 + \lambda_n, \lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots, \lambda_{n-1} - \lambda_n) \in C_{n,m}.$$

Conversely, we define $d_{n,m} : C_{n,m} \rightarrow I_{n,m}$ by sending $(a_0, a_1, \dots, a_{n-1})$ to partition $(a_1 + \dots + a_{n-1}, a_2 + \dots + a_{n-1}, \dots, a_{n-1}, 0, \dots)$.

Note that $|d_{n,m}(a_0, a_1, \dots, a_{n-1})| = \sum_i i a_i$. For $a = (a_0, a_1, \dots, a_{n-1}) \in C_{n,m}$, we say that $|d_{n,m}(a)| + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$ is the degree of a and write it $\deg(a)$. In particular, we will consider the subset $C_{n,m}^0 = \{a \in C_{n,m} \mid \deg(a) = 0 \pmod{n}\}$. Let $\rho_n : C_{n,m} \rightarrow C_{n,m}$ be the cyclic permutation $\rho_n(a_0, a_1, \dots, a_{n-1}) = (a_{n-1}, a_0, \dots, a_{n-2})$. We also define $\tau : C_{n,m}^0 \rightarrow C_{m,n}^0$ by

$$\tau(a) = \rho_m^{\frac{-|d_{n,m}(a)|}{n}}(w_{m,n}(d_{n,m}(a)^t)).$$

Then we have the following result which was proved in [52].

Theorem 3.2. *Let $\dot{\Lambda}_1, \dot{\Lambda}_2$ be the fundamental weights of sl_3 and let $\ddot{\Lambda}_1, \dots, \ddot{\Lambda}_8$ be the fundamental weights of sl_9 . Then the decomposition of $L_{sl_{27}}(1, 0)$ viewed as an $L_{sl_3}(9, 0) \otimes L_{sl_9}(3, 0)$ -module is as follows:*

$$L_{sl_{27}}(1, 0) = \oplus_{a \in C_{3,9}^0} L_{sl_3}(9, \widehat{a}) \otimes L_{sl_9}(3, \widehat{\tau(a)}),$$

where \widehat{a} denotes $a_1\dot{\Lambda}_1 + a_2\dot{\Lambda}_2$ and $\widehat{\tau(a)}$ denotes $\tau(a)_1\ddot{\Lambda}_1 + \dots + \tau(a)_8\ddot{\Lambda}_8$.

On the other hand, it is well-known that $L_{sl_3}(9, 0)$ has an extension vertex operator algebra $L_{E_6}(1, 0)$ (see [3, 36]). Moreover, it was shown in [36] that the vertex operator algebra $L_{E_6}(1, 0)$ viewed as an $L_{sl_3}(9, 0)$ -module has the following decomposition

$$\begin{aligned} L_{E_6}(1, 0) = & L_{sl_3}(9, 0) \oplus L_{sl_3}(9, 9\dot{\Lambda}_1) \oplus L_{sl_3}(9, 9\dot{\Lambda}_2) \oplus L_{sl_3}(9, \dot{\Lambda}_1 + 4\dot{\Lambda}_2) \\ & \oplus L_{sl_3}(9, 4\dot{\Lambda}_1 + \dot{\Lambda}_2) \oplus L_{sl_3}(9, 4\dot{\Lambda}_1 + 4\dot{\Lambda}_2). \end{aligned}$$

Furthermore, we have the following result which is a slight generalization of Theorem 3.8 of [3].

Theorem 3.3. *Let $(U^1, Y_1(\cdot, z))$, $(U^2, Y_2(\cdot, z))$ be extension vertex operator algebras of $L_{sl_3}(9, 0)$ such that $(U^1, Y_1(\cdot, z))$, $(U^2, Y_2(\cdot, z))$ are strongly regular and that U^1, U^2*

viewed as modules of $L_{sl_3}(9, 0)$ have the following decomposition

$$\begin{aligned} U^1 \cong U^2 \cong & L_{sl_3}(9, 0) \oplus L_{sl_3}(9, 9\Lambda_2) \oplus L_{sl_3}(9, 9\Lambda_1) \oplus L_{sl_3}(9, \Lambda_1 + 4\Lambda_2) \\ & \oplus L_{sl_3}(9, 4\Lambda_1 + \Lambda_2) \oplus L_{sl_3}(9, 4\Lambda_1 + 4\Lambda_2). \end{aligned}$$

Then there exists an isomorphism $\phi : U^1 \rightarrow U^2$ such that

$$\phi|_{L_{sl_3}(9,0)} = \text{id}, \quad \phi(Y_1(u^1, z)u^2) = Y_2(\phi(u^1), z)\phi(u^2), \text{ for any } u^1, u^2 \in U^1.$$

Proof: By assumption, $(U^1, Y_1(\cdot, z))$ is an extension vertex operator algebra of $L_{sl_3}(9, 0)$ such that $(U^1, Y_1(\cdot, z))$ is strongly regular and that U^1 viewed as a module of $L_{sl_3}(9, 0)$ has the following decomposition

$$\begin{aligned} U^1 \cong & L_{sl_3}(9, 0) \oplus L_{sl_3}(9, 9\Lambda_2) \oplus L_{sl_3}(9, 9\Lambda_1) \oplus L_{sl_3}(9, \Lambda_1 + 4\Lambda_2) \\ & \oplus L_{sl_3}(9, 4\Lambda_1 + \Lambda_2) \oplus L_{sl_3}(9, 4\Lambda_1 + 4\Lambda_2). \end{aligned}$$

It follows from Theorem 3.8 of [3] that there exists an isomorphism $\psi_1 : U^1 \rightarrow L_{E_6}(1, 0)$ such that

$$\psi_1(Y_1(u, z)v) = Y_{L_{E_6}(1,0)}(\psi_1(u), z)\psi_1(v)$$

for any $u, v \in U^1$, where $Y_{L_{E_6}(1,0)}(\cdot, z)$ denotes the vertex operator map of $L_{E_6}(1, 0)$. In particular, $\psi_1|_{L_{sl_3}(9,0)}$ is an automorphism of $L_{sl_3}(9, 0)$.

We next show that there exists an automorphism ψ_2 of $L_{E_6}(1, 0)$ such that $\psi_2 \circ \psi_1|_{L_{sl_3}(9,0)} = \text{id}$. It is good enough to show that any automorphism of $L_{sl_3}(9, 0)$ can be lifted to an automorphism of $L_{E_6}(1, 0)$. Note that any automorphism of sl_3 has the form $\varphi \exp(2\pi\sqrt{-1}h)$, where φ denotes the diagram automorphism of sl_3 and h is an element of a Cartan subalgebra of sl_3 (see [46]). Moreover, it was shown in subsection 2.5 of [49] that the diagram automorphism φ of sl_3 can be lifted to an automorphism of E_6 . Since for any h , $\exp(2\pi\sqrt{-1}h)$ is also an automorphism of E_6 , we then know that any automorphism of sl_3 can be lifted to an automorphism of E_6 . It follows that any automorphism of $L_{sl_3}(9, 0)$ can be lifted to an automorphism of $L_{E_6}(1, 0)$.

Similarly, there exists a vertex operator algebra isomorphism $\phi_2 : U^2 \rightarrow L_{E_6}(1, 0)$ such that $\phi_2|_{L_{sl_3}(9,0)} = \text{id}$. Hence, $\phi_2^{-1} \circ \psi_2 \circ \psi_1$ is the desired isomorphism. The proof is complete. \square

The next lemma can be proved by a direct calculation.

Lemma 3.4. *Let τ be the map defined as above. Then we have*

$$\begin{aligned}\tau((9, 0, 0)) &= (3, 0, 0, 0, 0, 0, 0, 0, 0), \\ \tau((0, 9, 0)) &= (0, 0, 0, 3, 0, 0, 0, 0, 0), \\ \tau((0, 0, 9)) &= (0, 0, 0, 0, 0, 0, 3, 0, 0), \\ \tau((4, 4, 1)) &= (0, 0, 0, 1, 0, 0, 0, 1, 1), \\ \tau((4, 1, 4)) &= (0, 1, 1, 0, 0, 0, 1, 0, 0), \\ \tau((1, 4, 4)) &= (1, 0, 0, 0, 1, 1, 0, 0, 0).\end{aligned}$$

Since the affine vertex operator algebra $L_{E_6}(1, 0)$ is self-dual, we then have the following result by Theorems 2.11, 3.2 and Lemma 3.4.

Theorem 3.5. *There is a vertex operator algebra structure on*

$$\begin{aligned}\widetilde{L_{sl_9}(3, 0)} &= L_{sl_9}(3, 0) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_3) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_6) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_2 + \ddot{\Lambda}_6) \\ &\quad \oplus L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_7 + \ddot{\Lambda}_8) \oplus L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_5)\end{aligned}$$

such that $\widetilde{L_{sl_9}(3, 0)}$ is an extension vertex operator algebra of $L_{sl_9}(3, 0)$ and strongly regular.

Moreover, by Theorems 2.12 and 3.3, we have:

Theorem 3.6. *Let $(U^1, Y_1(\cdot, z))$ and $(U^2, Y_2(\cdot, z))$ be two strongly regular vertex operator algebras satisfying the following conditions:*

- (1) U^1 and U^2 are extension vertex operator algebras of $L_{sl_9}(3, 0)$.
- (2) U^1 and U^2 viewed as $L_{sl_9}(3, 0)$ -modules are isomorphic to

$$\begin{aligned}L_{sl_9}(3, 0) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_3) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_6) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_2 + \ddot{\Lambda}_6) \\ \oplus L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_7 + \ddot{\Lambda}_8) \oplus L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_5).\end{aligned}$$

Then there exists a vertex operator algebra isomorphism $\phi^c : U^1 \rightarrow U^2$ such that $\phi^c|_{L_{sl_9}(3, 0)} = \text{id}$.

3.3. Lattice vertex operator algebras. In this subsection, we recall some facts about lattice vertex operator algebras from [6], [26] and [44]. Let L be a positive definite even lattice. We denote the \mathbb{Z} -bilinear form on L by $\langle \cdot, \cdot \rangle$. There is a canonical \mathbb{Z} -bilinear form c_0 on L defined as follows:

$$\begin{aligned}c_0 : L \times L &\rightarrow \mathbb{Z}/2\mathbb{Z} \\ (\alpha, \beta) &\mapsto \langle \alpha, \beta \rangle + 2\mathbb{Z}.\end{aligned}$$

Since L is an even lattice, the \mathbb{Z} -bilinear form c_0 is alternating. Thus there is a central extension \widehat{L} of L by the cyclic group $\langle \kappa \rangle$ of order 2 with generator κ , that is,

$$1 \rightarrow \langle \kappa \rangle \rightarrow \widehat{L} \xrightarrow{\pi} L \rightarrow 1,$$

such that the corresponding commutator map is c_0 (see [26]). We choose a section $e : L \rightarrow \widehat{L}$ such that $e_0 = 1$ and that the corresponding 2-cocycle $\epsilon_0 : L \times L \rightarrow \mathbb{Z}/2\mathbb{Z}$, which is defined by $e_\alpha e_\beta = \kappa^{\epsilon_0(\alpha, \beta)} e_{\alpha+\beta}$ for $\alpha, \beta \in L$, is a \mathbb{Z} -bilinear form satisfying the following condition:

$$\epsilon_0(\alpha, \alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle.$$

Hence, we have $\epsilon_0(\alpha, \beta) - \epsilon_0(\beta, \alpha) = c_0(\alpha, \beta)$ for $\alpha, \beta \in L$ (see [26]).

Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the \mathbb{Z} -bilinear form on L to \mathfrak{h} by \mathbb{C} -linearity. The corresponding affine Lie algebra is $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ with Lie brackets

$$\begin{aligned} [x(m), y(n)] &= \langle x, y \rangle m \delta_{m+n, 0} c, \\ [c, \widehat{\mathfrak{h}}] &= 0, \end{aligned}$$

for $x, y \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$, where $x(n)$ denotes $x \otimes t^n$. Set

$$\widehat{\mathfrak{h}}^- = \mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}].$$

Hence, $\widehat{\mathfrak{h}}^-$ is an abelian subalgebra of $\widehat{\mathfrak{h}}$. We then consider the induced $\widehat{\mathfrak{h}}$ -module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}c)} \mathbb{C} \cong S(\widehat{\mathfrak{h}}^-) \quad (\text{linearly}),$$

where $U(\cdot)$ denotes the universal enveloping algebra and $\mathbb{C}[t] \otimes \mathfrak{h}$ acts trivially on \mathbb{C} , c acts on \mathbb{C} as multiplication by 1.

Consider the \widehat{L} -module

$$\mathbb{C}\{L\} = \mathbb{C}[\widehat{L}] / \mathbb{C}[\widehat{L}](\kappa + 1),$$

where $\mathbb{C}[\cdot]$ denotes the group algebra. For $a \in \widehat{L}$, we use $\iota(a)$ to denote the image of a in $\mathbb{C}\{L\}$. Then the action of \widehat{L} on $\mathbb{C}\{L\}$ is given by

$$a \cdot \iota(b) = \iota(ab), \quad \kappa \cdot \iota(b) = -\iota(b)$$

for $a, b \in \widehat{L}$. For a formal variable z and an element $h \in \mathfrak{h}$, we define an operator $h(0)$ on $\mathbb{C}\{L\}$ by $h(0) \cdot \iota(a) = \langle h, \bar{a} \rangle \iota(a)$ and an action z^h on $\mathbb{C}\{L\}$ by $z^h \cdot \iota(a) = z^{\langle h, \bar{a} \rangle} \iota(a)$.

Set

$$V_L = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L\}.$$

Then \widehat{L} , $h(n)(n \neq 0)$, $h(0)$ and z^h act naturally on V_L by acting on either $M(1)$ or $\mathbb{C}\{L\}$ as indicated above. Denote $\iota(1)$ by $\mathbf{1}$ and set

$$\omega = \frac{1}{2} \sum_{i=1}^d h_i(-1)^2 \mathbf{1},$$

where h_1, \dots, h_d is an orthonormal basis of \mathfrak{h} . Then we know that $(V_L, Y(., z), \mathbf{1}, \omega)$ has a vertex operator algebra structure (see [6], [26]), the vertex operator $Y(., z)$ is determined by

$$\begin{aligned} Y(h(-1)\mathbf{1}, z) &= h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1} \quad (h \in \mathfrak{h}), \\ Y(a, z) &= E^-(\bar{a}, z) E^+(-\bar{a}, z) a z^{\bar{a}} \quad (a \in \widehat{L}), \end{aligned}$$

where

$$E^-(\bar{a}, z) = \exp\left(\sum_{n < 0} \frac{\bar{a}(n)}{n} z^{-n}\right), \quad E^+(\bar{a}, z) = \exp\left(\sum_{n > 0} \frac{\bar{a}(n)}{n} z^{-n}\right).$$

3.4. Automorphisms of mirror extension $\widetilde{L_{sl_9}(3, 0)}$. We now consider the root lattice of type A_8 and let L be the positive definite even lattice isomorphic to $A_8 \oplus A_8 \oplus A_8$. Let η_i , $i = 1, 2, 3$, be the natural inclusion of A_8 into the i -th summand of $L = A_8 \oplus A_8 \oplus A_8$.

Let $(A_8)_2 = \{\alpha \in A_8 \mid \langle \alpha, \alpha \rangle = 2\}$. Set $\tilde{h} = (\eta_1 + \eta_2 + \eta_3)(h)(-1) \cdot \mathbf{1} \in V_L$ for any $h \in A_8 \otimes_{\mathbb{Z}} \mathbb{C}$ and

$$E_\alpha = \iota(e_{\eta_1(\alpha)}) + \iota(e_{\eta_2(\alpha)}) + \iota(e_{\eta_3(\alpha)}) \in V_L \quad \text{for } \alpha \in (A_8)_2.$$

Then the vertex operator subalgebra $\langle S \rangle$ of V_L generated by

$$S = \{\tilde{h} \mid h \in A_8 \otimes_{\mathbb{Z}} \mathbb{C}\} \cup \{E_\alpha \mid \alpha \in (A_8)_2, \langle \alpha, \alpha \rangle = 2\}$$

is isomorphic to the affine VOA $L_{sl_9}(3, 0)$ (see [13, 26]). Moreover, it was proved in [26] that $\{\tilde{h} \mid h \in A_8 \otimes_{\mathbb{Z}} \mathbb{C}\}$ and $\{E_\alpha \mid \alpha \in (A_8)_2\}$ satisfy the following relations

$$\begin{aligned} [\tilde{h}, E_\alpha] &= \langle h, \alpha \rangle E_\alpha, \\ [E_\alpha, E_{-\alpha}] &= (-1)^{\epsilon_0(\alpha, -\alpha)} \tilde{\alpha}, \\ [E_\alpha, E_\beta] &= (-1)^{\epsilon_0(\alpha, \beta)} E_{\alpha+\beta}, \quad \text{if } \alpha + \beta \in (A_8)_2, \\ [E_\alpha, E_\beta] &= 0, \quad \text{otherwise.} \end{aligned}$$

Let Ω be the conformal element of $L_{sl_9}(3, 0)$. By the Sugawara construction, it is given by

$$\Omega = \frac{1}{2(3+9)} \left[\sum_{k=1}^8 \tilde{h}^k + \sum_{\alpha \in (A_8)_2} (E_\alpha)_{-1} (-E_{-\alpha}) \right],$$

where $\{h^1, \dots, h^8\}$ is an orthonormal basis of $A_8 \otimes_{\mathbb{Z}} \mathbb{C}$. Note that the dual vector of E_α is $-E_{-\alpha}$ since $\epsilon_0(\alpha, \alpha) = 1$.

Let $E = \{(\alpha, \alpha, \alpha) \mid \alpha \in A_8\}$ and $P = \{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in A_8, \alpha + \beta + \gamma = 0\}$. Then, by a direct calculation (see [9]), we have

$$\Omega = \omega_E + \frac{3}{4}\omega_P - \frac{1}{12} \sum_{\substack{\alpha \in (A_8)_2 \\ 1 \leq i < j \leq 3}} e_{\eta_i(\alpha) - \eta_j(\alpha)},$$

where ω_S denotes the conformal element of the lattice VOA V_S .

Let $\theta : \widehat{L} \rightarrow \widehat{L}$ be the automorphism of \widehat{L} defined by $\theta(a) = \kappa^{\epsilon_0(\bar{a}, \bar{a})} a^{-1}$. It was shown in [26] that θ induces an automorphism of V_L which is also denoted by θ . Explicitly, $\theta : V_L \rightarrow V_L$ is the linear map defined by

$$\theta(\alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes \iota(a)) = (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes \iota(\theta(a)).$$

In particular, we have $\theta(\iota(e_{(\alpha, 0, 0)})) = -\iota(e_{(-\alpha, 0, 0)})$, $\theta(\iota(e_{(0, \alpha, 0)})) = -\iota(e_{(0, -\alpha, 0)})$ and $\theta(\iota(e_{(0, 0, \alpha)})) = -\iota(e_{(0, 0, -\alpha)})$ for any $\alpha \in (A_8)_2$. Therefore, we have $\theta(\langle S \rangle) = \langle S \rangle$, that is, $\theta|_{\langle S \rangle}$ induces an automorphism of $L_{sl_9}(3, 0)$, which is also denoted by θ .

As a key result of this section, we shall show that the automorphism θ of $L_{sl_9}(3, 0)$ may be lifted to an automorphism of $\widetilde{L_{sl_9}(3, 0)}$.

First, we have the following general result which is a slight generalization of Proposition 3.2 of [55].

Theorem 3.7. *Let V be a strongly regular vertex operator algebra. Let g be an automorphism of V and U an extension vertex operator algebra of V . Assume further that V , g and U satisfy the following conditions:*

(1) *U viewed as a V -module has the decomposition*

$$U = M^0 \oplus M^1 \oplus \cdots \oplus M^k,$$

such that $M^0 = V, M^1, \dots, M^k$ are nonisomorphic irreducible V -modules. Moreover,

$$\{M^0, M^1, \dots, M^k\} = \{M^0 \circ g, M^1 \circ g, \dots, M^k \circ g\},$$

where $M^i \circ g$ denotes the V -module such that $M^i \circ g = M^i$ as vector space and the vertex operator $Y_{M^i \circ g}(v, z) = Y_{M^i}(g(v), z)$.

(2) *For any two strongly regular VOA structures $(U, Y_1(\cdot, z))$ and $(U, Y_2(\cdot, z))$ on $U = M^0 \oplus M^1 \oplus \cdots \oplus M^k$, there exists a VOA isomorphism Ψ from $(U, Y_1(\cdot, z))$ to $(U, Y_2(\cdot, z))$ such that $\Psi|_V = \text{id}$.*

Then there exists an automorphism \tilde{g} of U such that $\tilde{g}(V) = V$ and that $\tilde{g}|_V = g$.

Proof: The idea of the proof is similar to that in Theorem 2.1 of [56]. Let $\Psi : \{0, 1, \dots, k\} \rightarrow \{0, 1, \dots, k\}$ be the permutation such that

$$M^i \circ g \cong M^{\Psi(i)}, \quad i \in \{0, 1, \dots, k\}.$$

Fix V -module isomorphisms $\varphi_i : M^i \circ g \cong M^{\Psi(i)}$, $i \in \{0, 1, \dots, k\}$, such that $\varphi_0 = \text{id}$. Let $\psi_i : M^i \rightarrow M^i \circ g$, $i \in \{0, 1, \dots, k\}$, be the canonical linear maps such that

$$\psi_i(Y_{M^i}(g(v), z)w) = Y_{M^i \circ g}(v, z)\psi_i(w),$$

for any $v \in V$ and $w \in M^i$ (cf. [23]). In particular, we take $\psi_0 = g^{-1}$. We then define a linear isomorphism $\Phi : U \rightarrow U$ by

$$\Phi|_{M^i} = \varphi_i \circ \psi_i,$$

and a linear map $\tilde{Y}_U(\cdot, z)$ by

$$\begin{aligned} \tilde{Y}_U(\cdot, z) : U &\rightarrow (\text{End } U)[[z^{-1}, z]] \\ u &\mapsto \Phi^{-1}Y_U(\Phi(u), z)\Phi, \end{aligned}$$

where $u \in U$ and $Y_U(\cdot, z)$ denotes the vertex operator map of U . It is easy to verify that $(U, \tilde{Y}_U(\cdot, z))$ is also a vertex operator algebra. Moreover, we have $\Phi|_V = g^{-1}$; it implies $\tilde{Y}_U(u, z)v = Y_U(u, z)v$ for any $u, v \in V$. Hence, $(U, \tilde{Y}_U(\cdot, z))$ is also an extension vertex operator algebra of V . We next prove that $(U, \tilde{Y}_U(\cdot, z))$ is simple. Otherwise, assume that I is a proper ideal of $(U, \tilde{Y}_U(\cdot, z))$, it is clear that $\Phi(I)$ is a proper ideal of $(U, Y_U(\cdot, z))$, this is a contradiction. By Theorem 2.4, we know that $(U, \tilde{Y}_U(\cdot, z))$ is also a strongly regular vertex operator algebra. Then by assumption (2), there exists a linear isomorphism $\Psi : U \rightarrow U$ such that $\Psi|_V = \text{id}$ and that

$$\Psi(\tilde{Y}_U(u^1, z)u^2) = Y_U(\Psi(u^1), z)\Psi(u^2),$$

for any $u^1, u^2 \in U$. In particular, we have

$$\Psi(\Phi^{-1}Y_U(\Phi(u^1), z)\Phi(u^2)) = Y_U(\Psi(u^1), z)\Psi(u^2),$$

for any $u^1, u^2 \in U$. This implies that $\Psi \circ \Phi$ is an automorphism of U such that $\Psi \circ \Phi|_V = g^{-1}$. Therefore, $(\Psi \circ \Phi)^{-1}$ is the desired automorphism. The proof is complete. \square

To prove that the automorphism θ of $L_{sl_9}(3, 0)$ can be lifted to an automorphism of $\widetilde{L_{sl_9}(3, 0)}$, we need the following:

Lemma 3.8. *Let $L_{sl_9}(3, L(\ddot{\Lambda}))$ be an irreducible module $L_{sl_9}(3, 0)$. Then we have*

$$L_{sl_9}(3, L(\ddot{\Lambda})) \circ \theta \cong L_{sl_9}(3, L(\ddot{\Lambda})^*) \cong L_{sl_9}(3, L(\ddot{\Lambda}))',$$

where $L(\ddot{\Lambda})^*$ denotes the dual module of $L(\ddot{\Lambda})$.

Proof: For any $x \in sl_9$, let $Y_{L_{sl_9}(3, L(\ddot{\Lambda})) \circ \theta}(x, z) = \sum_{n \in \mathbb{Z}} x^\theta(n) z^{-n-1}$. By the definition of $L_{sl_9}(3, L(\ddot{\Lambda})) \circ \theta$, we have

$$x^\theta(n)v = \theta(x)(n)v$$

for any $x \in sl_9$, $v \in L_{sl_9}(3, L(\ddot{\Lambda}))$ and $n \in \mathbb{Z}$. In particular, we have

$$\tilde{h}^\theta(n)v = -\tilde{h}(n)v, \quad E_\alpha^\theta(n)v = -E_{-\alpha}(n)v, \quad (3.1)$$

for any $h \in A_8 \otimes_{\mathbb{Z}} \mathbb{C}$ and $\alpha \in (A_8)_2$.

For any irreducible $L_{sl_9}(3, 0)$ -module M , set

$$\Omega(M) = \{v \in M \mid x(n)v = 0 \text{ for } x \in sl_9, n \geq 1\}.$$

Then we know that $\Omega(M)$ is an irreducible sl_9 -module (see [27]). Since $L_{sl_9}(3, L(\ddot{\Lambda}))$ is irreducible, we know that $L_{sl_9}(3, L(\ddot{\Lambda})) \circ \theta$ is also an irreducible $L_{sl_9}(3, 0)$ -module. In particular, $\Omega(L_{sl_9}(3, L(\ddot{\Lambda})) \circ \theta)$ is an irreducible sl_9 -module. By formula (3.1), we know that $\Omega(L_{sl_9}(3, L(\ddot{\Lambda})) \circ \theta)$ is a lowest weight sl_9 -module with lowest weight $-\ddot{\Lambda}$. Hence, we have $\Omega(L_{sl_9}(3, L(\ddot{\Lambda})) \circ \theta) \cong L(\ddot{\Lambda})^*$. In particular, $L_{sl_9}(3, L(\ddot{\Lambda})) \circ \theta \cong L_{sl_9}(3, L(\ddot{\Lambda})^*)$.

On the other hand, by the definition of $L_{sl_9}(3, L(\ddot{\Lambda}))'$, we have

$$\langle x(n)f, v \rangle = \langle f, -x(-n)v \rangle, \quad (3.2)$$

for any $x \in sl_9$, $f \in L_{sl_9}(3, L(\ddot{\Lambda}))'$ and $n \in \mathbb{Z}$. Since $L_{sl_9}(3, L(\ddot{\Lambda}))'$ is an irreducible $L_{sl_9}(3, 0)$ -module, we know that $\Omega(L_{sl_9}(3, L(\ddot{\Lambda}))')$ viewed as a vector space is equal to $L(\ddot{\Lambda})^*$ and is an irreducible sl_9 -module. It follows from the formula (3.2) that $\Omega(L_{sl_9}(3, L(\ddot{\Lambda}))')$ viewed as an irreducible sl_9 -module is equal to $L(\ddot{\Lambda})^*$. Hence, we have $L_{sl_9}(3, L(\ddot{\Lambda})^*) \cong L_{sl_9}(3, L(\ddot{\Lambda}))'$. The proof is complete. \square

Combining Theorems 3.6, 3.7 and Lemma 3.8, we have:

Theorem 3.9. *There exists an automorphism $\tilde{\theta}$ of $\widetilde{L_{sl_9}(3, 0)}$ such that $\tilde{\theta}(L_{sl_9}(3, 0)) = L_{sl_9}(3, 0)$ and that $\tilde{\theta}|_{L_{sl_9}(3, 0)} = \theta$.*

Proof: By Theorems 3.6, 3.7, it is sufficient to verify that

$$\begin{aligned} & \{L_{sl_9}(3, 0), L_{sl_9}(3, 3\ddot{\Lambda}_3), L_{sl_9}(3, 3\ddot{\Lambda}_6), L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_2 + \ddot{\Lambda}_6), L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_7 + \ddot{\Lambda}_8), L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_5)\} \\ &= \{L_{sl_9}(3, 0) \circ \theta, L_{sl_9}(3, 3\ddot{\Lambda}_3) \circ \theta, L_{sl_9}(3, 3\ddot{\Lambda}_6) \circ \theta, L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_2 + \ddot{\Lambda}_6) \circ \theta, \\ & \quad L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_7 + \ddot{\Lambda}_8) \circ \theta, L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_5) \circ \theta\}. \end{aligned} \quad (3.3)$$

Note that $\widetilde{L_{sl_9}(3, 0)}$ is self-dual, the formula (3.3) follows immediately from Lemma 3.8. The proof is complete. \square

4. HOLOMORPHIC VOA OF CENTRAL CHARGE 24 WITH LIE ALGEBRA $A_{8,3}A_{2,1}^2$

In this section, we shall study the holomorphic vertex operator algebra of central charge 24 with Lie algebra $A_{8,3}A_{2,1}^2$. Recall from [41] that there exists a holomorphic vertex operator algebra U such that the central charge of U is 24 and the Lie algebra U_1 is isomorphic to $A_{8,3}A_{2,1}^2$. Moreover, it was also proved in [41] that U is strongly regular. We shall study the structure of U in this section.

4.1. Vertex operator subalgebras of U . In this subsection, we shall study some vertex operator subalgebras of U . Note that U contains a vertex operator subalgebra isomorphic to $L_{sl_9}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Set $V^2 = C_U(L_{sl_3}(1,0) \otimes L_{sl_3}(1,0))$. Our goal in this subsection is to determine the vertex operator subalgebra V^2 . First, we have the following:

Lemma 4.1. *The vertex operator algebra V^2 is strongly regular.*

Proof: Since U is of CFT-type, it follows that V^2 is of CFT-type. Note also that V^2 contains the vertex operator subalgebra $L_{sl_9}(3,0)$, hence V^2 is an extension vertex operator algebra of $L_{sl_9}(3,0)$. By Theorem 2.4, to prove that V^2 is strongly regular, it is sufficient to show that V^2 is simple. Note that $L_{sl_3}(1,0)$ is isomorphic to the lattice vertex operator algebra V_{A_2} , where A_2 denotes the root lattice of type A_2 . Let G be the dual group of $(A_2 \oplus A_2)^*/(A_2 \oplus A_2)$. Then we know that there is an action of G on U such that $U^G = V^2 \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. It follows that $V^2 \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ is simple (see [22]). This implies that V^2 is simple. The proof is complete. \square

We next determine the global dimension of V^2 . First, we have the following result which was proved in [38].

Theorem 4.2. *Let V be a strongly regular vertex operator algebra and W a strongly regular vertex operator subalgebra of V . Suppose also that the commutant $C_V(W)$ of W is strongly regular and satisfies $C_V(C_V(W)) = W$. Then all the irreducible W -modules appear in some simple V -module.*

To apply Theorem 4.2, we also need to determine the commutant $C_U(V^2)$ of V^2 . Let $\dot{\Lambda}_1, \dot{\Lambda}_2$ be the fundamental weights of sl_3 . It is well-known that $L_{sl_3}(1,0)$ has three nonisomorphic irreducible modules $L_{sl_3}(1,0)$, $L_{sl_3}(1, \dot{\Lambda}_1)$, $L_{sl_3}(1, \dot{\Lambda}_2)$. The conformal weights of $L_{sl_3}(1,0)$, $L_{sl_3}(1, \dot{\Lambda}_1)$, $L_{sl_3}(1, \dot{\Lambda}_2)$ are equal to 0, $1/3$, $1/3$, respectively. Moreover, $L_{sl_3}(1, \dot{\Lambda}_1)$, $L_{sl_3}(1, \dot{\Lambda}_2)$ are simple current $L_{sl_3}(1,0)$ -modules such that $L_{sl_3}(1, \dot{\Lambda}_1) \times L_{sl_3}(1, \dot{\Lambda}_1) \cong L_{sl_3}(1, \dot{\Lambda}_2)$ (see [15], [47]). In particular, the fusion ring of $L_{sl_3}(1,0)$ is isomorphic to \mathbb{Z}_3 .

Lemma 4.3. *The commutant $C_U(V^2)$ of V^2 is equal to $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$.*

Proof: Note that $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ is a vertex operator subalgebra of $C_U(V^2)$. Hence, $C_U(V^2)$ is an extension vertex operator algebra of $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Comparing the conformal weights of irreducible $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -modules, we know that $C_U(V^2)$ is equal to $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. \square

Hence, by Theorems 2.2, 2.10, 4.2 and Lemmas 4.1, 4.3, we have the following:

Theorem 4.4. (1) *All the irreducible V^2 -modules appear in U . Moreover, there are 9 nonisomorphic irreducible V^2 -modules.*

(2) *All the irreducible V^2 -modules are simple current modules. In particular, the fusion ring of V^2 is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $\text{Glob } V^2 = 9$.*

To determine the vertex operator algebra structure of V^2 , note first that V^2 gives rise to a modular invariant of $L_{sl_9}(3,0)$ by Theorem 2.6. On the other hand, the modular invariants of the affine vertex operator algebra $L_{sl_9}(3,0)$ have been classified in [30]. To describe the result, we need to recall some notations from [30]. Recall that irreducible $L_{sl_9}(3,0)$ -modules are parameterized by $C_{9,3}$. For any $a = (a_0, a_1, \dots, a_8) \in C_{9,3}$, we use Z_a to denote the trace function associated to the irreducible $L_{sl_9}(3,0)$ -module $L_{sl_9}(3, a_1\check{\Lambda}_1 + \dots + a_8\check{\Lambda}_8)$ and $\langle Z_a \rangle$ to denote $\sum_{j=1}^3 Z_{\rho_9^{3j}(a)}$, where $\rho_9 : C_{9,3} \rightarrow C_{9,3}$ is the map defined in subsection 2.4. Let $C : C_{9,3} \rightarrow C_{9,3}$ be the map defined by $C((a_0, a_1, a_2, \dots, a_7, a_8)) = (a_0, a_8, a_7, \dots, a_2, a_1)$. We then define a matrix \mathcal{C} by $\mathcal{C}_{a,b} = \delta_{b,C(a)}$. For a modular invariant \mathcal{M} of the affine vertex operator algebra $L_{sl_9}(3,0)$, we define the *conjugate modular invariant* of \mathcal{M} to be the matrix $\mathcal{C}\mathcal{M}$ (cf. [30]).

Theorem 4.5. *Any modular invariant of the affine vertex operator algebra $L_{sl_9}(3,0)$ is equal to one of the following modular invariants or their conjugate modular invariants:*

$\mathcal{A}_{a,b} = \delta_{a,b}$	\mathcal{A}
$\mathcal{D}_{a,b}^{(1)} = \sum_{j=1}^9 \delta^9(d_{9,3}(a) + 6j) \delta_{b, \rho_9^j(a)}$	$\mathcal{D}^{(1)}$
$\mathcal{D}_{a,b}^{(3)} = \sum_{j=1}^3 \delta^3(d_{9,3}(a) + 18j) \delta_{b, \rho_9^{3j}(a)}$	$\mathcal{D}^{(3)}$
$\mathcal{D}_{a,b}^{(9)} = \delta^1(d_{9,3}(a) + 54) \delta_{b, \rho_9^9(a)}$	$\mathcal{D}^{(9)}$
$\sum_{i=0}^2 (\langle Z_{\rho_9^i(2,0,0,1,0,0,0,0,0)} \rangle ^2 + \langle Z_{\rho_9^i(3,0,0,0,0,0,0,0,0)} \rangle ^2 + \langle Z_{\rho_9^i(2,0,0,0,0,0,1,0,0)} \rangle ^2 + \langle Z_{\rho_9^i(1,1,0,0,0,1,0,0,0)} \rangle ^2 + \langle Z_{\rho_9^i(1,0,1,0,1,0,0,0,0)} \rangle ^2 + 2 \langle Z_{\rho_9^i(1,0,0,1,0,0,1,0,0)} \rangle ^2 + \langle Z_{\rho_9^i(0,0,1,1,1,0,0,0,0)} \rangle \overline{\langle Z_{\rho_9^i(1,0,0,1,0,0,1,0,0)} \rangle} + \overline{\langle Z_{\rho_9^i(1,0,0,1,0,0,1,0,0)} \rangle} \langle Z_{\rho_9^i(1,0,0,1,0,0,1,0,0)} \rangle + \langle Z_{\rho_9^i(0,0,1,1,1,0,0,0,0)} \rangle \overline{\langle Z_{\rho_9^i(1,0,0,1,0,0,1,0,0)} \rangle})$	\mathcal{E}
$\sum_{i=0}^2 (\langle Z_{\rho_9^i(1,0,0,0,1,1,0,0,0)} \rangle + \langle Z_{\rho_9^i(3,0,0,0,0,0,0,0,0)} \rangle ^2 + 2 \langle Z_{\rho_9^i(1,0,1,0,0,0,0,1,0)} \rangle ^2)$	\mathcal{E}'
$ \langle Z_{(1,0,0,0,1,1,0,0,0)} \rangle + \langle Z_{(3,0,0,0,0,0,0,0,0)} \rangle ^2 + \langle Z_{(1,0,1,0,0,0,0,1,0)} \rangle + \langle Z_{(1,0,1,0,1,0,0,0,0)} \rangle ^2 + \langle Z_{(0,1,0,1,0,1,0,0,0)} \rangle \overline{\langle \sum_{i=1}^2 \langle Z_{\rho_9^i(3,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{\rho_9^i(1,0,0,0,1,1,0,0,0)} \rangle \rangle} + (\sum_{i=1}^2 \langle Z_{\rho_9^i(3,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{\rho_9^i(1,0,0,0,1,1,0,0,0)} \rangle) \overline{\langle Z_{(0,1,0,1,0,1,0,0,0)} \rangle}$	\mathcal{E}''

where $a, b \in C_{9,3}$, $\delta^x(y) = 1$ if $y/x \in \mathbb{Z}$ and $\delta^x(y) = 0$ if $y/x \notin \mathbb{Z}$. Here, the modular invariants of types $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ should be interpreted similarly as the modular invariants in Subsection 2.4.

To determine the modular invariant corresponding to V^2 , we need the following:

Lemma 4.6. *The modular invariant corresponding to V^2 is not equal to the modular invariant of type \mathcal{E}'' , \mathcal{CE}' or \mathcal{CE}'' .*

Proof: By the definition of conjugate modular invariant, the modular invariant of type \mathcal{CE}' is equal to

$$\begin{aligned} & |\langle Z_{(3,0,0,0,0,0,0,0,0)} \rangle + \langle Z_{(1,0,0,0,1,1,0,0,0)} \rangle|^2 + |\langle Z_{(1,0,1,0,0,0,0,1,0)} \rangle|^2 \\ & + (\langle Z_{(0,0,3,0,0,0,0,0,0)} \rangle + \langle Z_{(0,0,1,0,0,0,1,1,0)} \rangle) \overline{(\langle Z_{(0,3,0,0,0,0,0,0,0)} \rangle + \langle Z_{(0,1,0,0,0,1,1,0,0)} \rangle)} \\ & + (\langle Z_{(0,3,0,0,0,0,0,0,0)} \rangle + \langle Z_{(0,1,0,0,0,1,1,0,0)} \rangle) \overline{(\langle Z_{(0,0,3,0,0,0,0,0,0)} \rangle + \langle Z_{(0,0,1,0,0,0,1,1,0)} \rangle)} \\ & + \langle Z_{(1,0,1,0,1,0,0,0,0)} \rangle \overline{\langle Z_{(0,1,0,1,0,0,0,0,1)} \rangle} + \langle Z_{(0,1,0,1,0,0,0,0,1)} \rangle \overline{\langle Z_{(1,0,1,0,1,0,0,0,0)} \rangle}. \end{aligned}$$

Note that the modular invariant of type \mathcal{CE}' contains $Z_{(0,3,0,0,0,0,0,0,0)} \overline{Z_{(0,0,3,0,0,0,0,0,0)}}$, but does not contain the term $Z_{(0,3,0,0,0,0,0,0,0)} \overline{Z_{(0,3,0,0,0,0,0,0,0)}}$, it follows from Lemma 2.7 that the modular invariant of type \mathcal{CE}' cannot be realized by extension vertex operator algebra.

Note also that the map $C : C_{9,3} \rightarrow C_{9,3}$ maps $\{\rho_9^i((1, 0, 1, 0, 1, 0, 0, 0, 0)) | 0 \leq i \leq 8\}$ to itself, and that

$$\{\rho_9^i((1, 0, 1, 0, 1, 0, 0, 0, 0)) | 0 \leq i \leq 8\} \cap \{\rho_9^i((1, 0, 0, 0, 1, 1, 0, 0, 0)) | 0 \leq i \leq 8\} = \emptyset.$$

It follows that the modular invariants of types \mathcal{E}'' and \mathcal{CE}'' contain

$$Z_{\rho_9^j(0,1,0,1,0,1,0,0,0)} \overline{Z_{(0,1,0,0,0,1,1,0,0)}}$$

for some j , but does not contain the term $Z_{(0,1,0,0,0,1,1,0,0)} \overline{Z_{(0,1,0,0,0,1,1,0,0)}}$. By Lemma 2.7, the modular invariants of types \mathcal{E}'' and \mathcal{CE}'' cannot be realized by extension vertex operator algebras. The proof is complete. \square

Combining Lemmas 4.1, 4.6 and Theorem 4.5, we have:

Theorem 4.7. *The vertex operator algebra V^2 is isomorphic to $\widetilde{L_{sl_9}(3,0)}$.*

Proof: The idea is to show that the modular invariant associated to V^2 is equal to the modular invariant of type \mathcal{E}' . Note first that the modular invariant associated to V^2 cannot be equal to the modular invariants of types \mathcal{A} , $\mathcal{D}^{(1)}$, $\mathcal{D}^{(3)}$, $\mathcal{D}^{(9)}$, \mathcal{E} or their conjugate modular invariants. Otherwise, V^2 must be a simple current extension of $L_{sl_9}(3,0)$. However, there are only three simple current $L_{sl_9}(3,0)$ -modules $L_{sl_9}(3,0)$, $L_{sl_9}(3,3\ddot{\Lambda}_3)$, $L_{sl_9}(3,3\ddot{\Lambda}_6)$ that have integral conformal weights. This implies the global dimension of $L_{sl_9}(3,0)$ must be less than 81 by Theorem 2.3. Since $L_{sl_9}(3,0)$ has more than 81 nonisomorphic irreducible modules, this is a contradiction by Theorem 2.2. By Lemma 4.6, the modular invariant associated to V^2 must be equal to the modular

invariant of type \mathcal{E}' . It follows immediately that V^2 viewed as a module of $L_{sl_9}(3, 0)$ is isomorphic to

$$\begin{aligned} & L_{sl_9}(3, 0) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_3) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_6) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_2 + \ddot{\Lambda}_6) \\ & \oplus L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_7 + \ddot{\Lambda}_8) \oplus L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_5). \end{aligned}$$

By Theorem 3.6, we know that V^2 is isomorphic to $\widetilde{L_{sl_9}(3, 0)}$. The proof is complete. \square

Since the modular invariant associated to V^2 is equal to the modular invariant of type \mathcal{E}' , we can obtain the following results immediately.

Theorem 4.8. (1) *There exists a unique $\widetilde{L_{sl_9}(3, 0)}$ -module structure on each of the following $L_{sl_9}(3, 0)$ -modules*

$$\begin{aligned} \widetilde{L_{sl_9}(3, 0)} &= L_{sl_9}(3, 0) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_3) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_6) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_2 + \ddot{\Lambda}_6) \\ &\quad \oplus L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_7 + \ddot{\Lambda}_8) \oplus L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_5), \\ \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} &= L_{sl_9}(3, 3\ddot{\Lambda}_1) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_4) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_7) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_5 + \ddot{\Lambda}_6) \\ &\quad \oplus L_{sl_9}(3, \ddot{\Lambda}_2 + \ddot{\Lambda}_3 + \ddot{\Lambda}_7) \oplus L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_8), \\ \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)} &= L_{sl_9}(3, 3\ddot{\Lambda}_2) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_5) \oplus L_{sl_9}(3, 3\ddot{\Lambda}_8) \oplus L_{sl_9}(3, \ddot{\Lambda}_2 + \ddot{\Lambda}_6 + \ddot{\Lambda}_7) \\ &\quad \oplus L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_4 + \ddot{\Lambda}_8) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_5). \end{aligned}$$

(2) *There exist two nonisomorphic $\widetilde{L_{sl_9}(3, 0)}$ -module structures on each of the following $L_{sl_9}(3, 0)$ -modules*

$$\begin{aligned} & L_{sl_9}(3, \ddot{\Lambda}_2 + \ddot{\Lambda}_7) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_3 + \ddot{\Lambda}_5) \oplus L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_6 + \ddot{\Lambda}_8), \\ & L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_3 + \ddot{\Lambda}_8) \oplus L_{sl_9}(3, \ddot{\Lambda}_2 + \ddot{\Lambda}_4 + \ddot{\Lambda}_6) \oplus L_{sl_9}(3, \ddot{\Lambda}_5 + \ddot{\Lambda}_7), \\ & L_{sl_9}(3, \ddot{\Lambda}_2 + \ddot{\Lambda}_4) \oplus L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_5 + \ddot{\Lambda}_7) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_6 + \ddot{\Lambda}_8). \end{aligned}$$

4.2. Fusion ring of mirror extension $\widetilde{L_{sl_9}(3, 0)}$. In Subsection 4.1, we already knew that the fusion ring of $\widetilde{L_{sl_9}(3, 0)}$ is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. In this subsection, we shall determine the fusion ring of $\widetilde{L_{sl_9}(3, 0)}$ explicitly. First, we need to recall some facts about the Weyl group of sl_{n+1} . Consider the $(n+1)$ -dimensional euclidean space \mathbb{R}^{n+1} . Let $\epsilon_1, \dots, \epsilon_{n+1}$ be the standard basis of \mathbb{R}^{n+1} . It is well-known that

$$\{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n+1\}$$

forms a root system of type A_n and that the fundamental weights are given by

$$\ddot{\Lambda}_i = \frac{1}{n+1}((n+1-i)(\epsilon_1 + \dots + \epsilon_i) - i(\epsilon_i + \dots + \epsilon_{n+1})),$$

$1 \leq i \leq n$ (see [34]). It is also known that the Weyl group of the root system of type A_n is isomorphic to the permutation group S_{n+1} . In particular, the reflection associated to the root $\epsilon_i - \epsilon_j$ corresponds to the permutation (i, j) (see [34]). The longest element of the Weyl group of root system of type A_n was also determined in [7], [34]. In particular, we have:

Lemma 4.9. *The longest element w_0 of the Weyl group of root system of type A_9 is equal to the permutation*

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1. \end{array}$$

In particular, we have $w_0(\ddot{\Lambda}_i) = -\ddot{\Lambda}_{9-i}$ for $1 \leq i \leq 8$.

We now let $L(\ddot{\Lambda})$ be the irreducible highest weight module of sl_9 with highest weight $\ddot{\Lambda}$. It is well-known that the dual module $L(\ddot{\Lambda})^*$ of $L(\ddot{\Lambda})$ is isomorphic to $L(-w_0(\ddot{\Lambda}))$ (see [7], [33]). Hence, we have the following:

Lemma 4.10. *Let $L(\ddot{\Lambda})$ be the irreducible highest weight module of sl_9 with highest weight $\ddot{\Lambda}$. Then we have $L_{sl_9}(3, L(\ddot{\Lambda}))' \cong L_{sl_9}(3, L(-w_0(\ddot{\Lambda})))$.*

We now turn back to determine the fusion ring of $\widetilde{L_{sl_9}(3, 0)}$. Recall that there exist two nonisomorphic $\widetilde{L_{sl_9}(3, 0)}$ -module structures on

$$L_{sl_9}(3, \ddot{\Lambda}_2 + \ddot{\Lambda}_7) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_3 + \ddot{\Lambda}_5) \oplus L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_6 + \ddot{\Lambda}_8).$$

We denote them by τ_1 and τ_2 , respectively. Then we have:

Theorem 4.11. *Let $\tau_1, \tau_2, \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)}, \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)}$ be the irreducible $\widetilde{L_{sl_9}(3, 0)}$ -modules defined as before. Then we have*

$$\begin{aligned} \tau_1 \times \tau_1 &\cong \tau_2, \\ \tau_1 \times \tau_2 &\cong \widetilde{L_{sl_9}(3, 0)}, \\ \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \times \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} &\cong \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)}, \\ \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \times \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)} &\cong \widetilde{L_{sl_9}(3, 0)}. \end{aligned}$$

Moreover, $\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \times \tau_1$ and $\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \times \tau_2$ viewed as $L_{sl_9}(3, 0)$ -modules are isomorphic and $L_{sl_9}(3, 3\ddot{\Lambda}_2) \times \tau_1$ and $L_{sl_9}(3, 3\ddot{\Lambda}_2) \times \tau_2$ viewed as $L_{sl_9}(3, 0)$ -modules are isomorphic.

Proof: Note that $L_{sl_9}(3, 3\ddot{\Lambda}_1)$ and $L_{sl_9}(3, 3\ddot{\Lambda}_2)$ are simple current $L_{sl_9}(3, 0)$ -modules. Moreover, we have the following result, which was proved in Corollary 2.27 of [47],

$$L_{sl_9}(3, 3\ddot{\Lambda}_1) \times L_{sl_9}(3, 3\ddot{\Lambda}_1) \cong L_{sl_9}(3, 3\ddot{\Lambda}_2), \quad L_{sl_9}(3, 3\ddot{\Lambda}_1) \times L_{sl_9}(3, 3\ddot{\Lambda}_2) \cong L_{sl_9}(3, 0).$$

It follows immediately that

$$\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \times \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \cong \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)}, \quad \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \times \widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)} \cong \widetilde{L_{sl_9}(3, 0)}.$$

We next show that $\tau_1 \times \tau_1 \cong \tau_2$. Recall that the fusion ring of $\widetilde{L_{sl_9}(3, 0)}$ is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, it is good enough to show that $\tau'_1 \cong \tau_2$. Note that τ_1 viewed as an $L_{sl_9}(3, 0)$ -module is isomorphic to

$$L_{sl_9}(3, \ddot{\Lambda}_2 + \ddot{\Lambda}_7) \oplus L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_3 + \ddot{\Lambda}_5) \oplus L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_6 + \ddot{\Lambda}_8).$$

It follows immediately from Lemmas 4.9, 4.10 that $\tau'_1 \cong \tau_1$ or τ_2 , which forces $\tau'_1 \cong \tau_2$. Hence, we have

$$\begin{aligned} \tau_1 \times \tau_1 &\cong \tau_2, \\ \tau_1 \times \tau_2 &\cong \widetilde{L_{sl_9}(3, 0)}. \end{aligned}$$

The last statement follows from the fact that $L_{sl_9}(3, 3\ddot{\Lambda}_1)$ and $L_{sl_9}(3, 3\ddot{\Lambda}_2)$ are simple current $L_{sl_9}(3, 0)$ -modules. The proof is complete. \square

4.3. Uniqueness of holomorphic VOA of central charge 24 with Lie algebra $A_{8,3}A_{2,1}^2$. In this subsection, we shall show that if \tilde{U} is a holomorphic vertex operator algebra such that the central charge of \tilde{U} is 24 and the Lie algebra \tilde{U}_1 is isomorphic to $A_{8,3}A_{2,1}^2$, then \tilde{U} is isomorphic to U . As an application, we shall construct some special automorphism of U .

By the similar discussion as above, we know that $C_{\tilde{U}}(L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0))$ is also isomorphic to $\widetilde{L_{sl_9}(3, 0)}$. Hence, \tilde{U} and U are extension vertex operator algebras of $\widetilde{L_{sl_9}(3, 0)} \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$. To determine the decompositions of \tilde{U} and U viewed as $L_{sl_9}(3, 0) \otimes L_{sl_3}(1, 0) \otimes L_{sl_3}(1, 0)$ -modules, we need the following:

Lemma 4.12. *Modulo integers, the conformal weights of irreducible $\widetilde{L_{sl_9}(3, 0)}$ -modules are as follows:*

$\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)}$	$\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)}$	τ_1	τ_2	$\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \times \tau_1$	$\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_1)} \times \tau_2$	$\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)} \times \tau_1$	$\widetilde{L_{sl_9}(3, 3\ddot{\Lambda}_2)} \times \tau_2$
$\frac{4}{3}$	$\frac{7}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$

The following result follows immediately from Theorems 2.10, 4.11 and Lemma 4.12.

$$\begin{aligned}
& \widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \oplus \widetilde{L_{sl_9}(3,3\check{\Lambda}_1)} \otimes L_{sl_3}(1,\check{\Lambda}_2) \otimes L_{sl_3}(1,\check{\Lambda}_1) \oplus \widetilde{L_{sl_9}(3,3\check{\Lambda}_2)} \otimes L_{sl_3}(1,\check{\Lambda}_1) \otimes L_{sl_3}(1,\check{\Lambda}_2) \\
& \oplus \tau_1 \otimes L_{sl_3}(1,\check{\Lambda}_2) \otimes L_{sl_3}(1,\check{\Lambda}_2) \oplus \tau_2 \otimes L_{sl_3}(1,\check{\Lambda}_1) \otimes L_{sl_3}(1,\check{\Lambda}_1) \oplus (\widetilde{L_{sl_9}(3,3\check{\Lambda}_1)} \times \tau_1) \otimes L_{sl_3}(1,\check{\Lambda}_1) \otimes L_{sl_3}(1,0) \\
& \oplus (\widetilde{L_{sl_9}(3,3\check{\Lambda}_2)} \times \tau_1) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,\check{\Lambda}_1) \oplus (\widetilde{L_{sl_9}(3,3\check{\Lambda}_1)} \times \tau_2) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,\check{\Lambda}_2) \\
& \oplus (\widetilde{L_{sl_9}(3,3\check{\Lambda}_2)} \times \tau_2) \otimes L_{sl_3}(1,\check{\Lambda}_2) \otimes L_{sl_3}(1,0) = W^7,
\end{aligned}$$

$$\begin{aligned}
& \widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \oplus \widetilde{L_{sl_9}(3,3\check{\Lambda}_1)} \otimes L_{sl_3}(1,\check{\Lambda}_2) \otimes L_{sl_3}(1,\check{\Lambda}_1) \oplus \widetilde{L_{sl_9}(3,3\check{\Lambda}_2)} \otimes L_{sl_3}(1,\check{\Lambda}_1) \otimes L_{sl_3}(1,\check{\Lambda}_2) \\
& \oplus \tau_1 \otimes L_{sl_3}(1,\check{\Lambda}_1) \otimes L_{sl_3}(1,\check{\Lambda}_1) \oplus \tau_2 \otimes L_{sl_3}(1,\check{\Lambda}_2) \otimes L_{sl_3}(1,\check{\Lambda}_2) \oplus (\widetilde{L_{sl_9}(3,3\check{\Lambda}_1)} \times \tau_1) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,\check{\Lambda}_2) \\
& \oplus (\widetilde{L_{sl_9}(3,3\check{\Lambda}_2)} \times \tau_1) \otimes L_{sl_3}(1,\check{\Lambda}_2) \otimes L_{sl_3}(1,0) \oplus (\widetilde{L_{sl_9}(3,3\check{\Lambda}_1)} \times \tau_2) \otimes L_{sl_3}(1,\check{\Lambda}_1) \otimes L_{sl_3}(1,0) \\
& \oplus (\widetilde{L_{sl_9}(3,3\check{\Lambda}_2)} \times \tau_2) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,\check{\Lambda}_1) = W^8.
\end{aligned}$$

To prove the uniqueness of holomorphic VOA of central charge 24 with Lie algebra $A_{8,3}A_{2,1}^2$, we also need the following:

Proposition 4.14. *Let V be a strongly regular vertex operator algebra and g an automorphism of V . Let V^1, V^2 be extension vertex operator algebras of V . Assume that V, g, V^1, V^2 satisfy the following conditions:*

- (1) *There exists a unique vertex operator algebra structure on V^2 such that V^2 is an extension vertex operator algebra of V .*
- (2) *$V^1 \circ g$ viewed as a V -module is isomorphic to V^2 . Moreover, there exists a V -module isomorphism $\phi : V^1 \circ g \rightarrow V^2$ such that $\phi|_V = g^{-1}$.*

Then the vertex operator algebra V^1 is isomorphic to V^2 .

Proof: Let Y_1 and Y_2 be the vertex operator maps of V^1 and V^2 , respectively. By assumption, there exists a V -module isomorphism ϕ from $V^1 \circ g$ to V^2 such that $\phi|_V = g^{-1}$. In particular, for any $u, v \in V$, we have

$$\phi(Y_1(\phi^{-1}(u), z)v) = Y_2(u, z)\phi(v).$$

Define a linear map

$$\begin{aligned}
Y_2^g : V^2 &\rightarrow \text{End}(V^2)[[z^{-1}, z]] \\
v &\mapsto \phi Y_1(\phi^{-1}(v), z)\phi^{-1}.
\end{aligned}$$

It is easy to show that (V^2, Y_2^g) is a vertex operator algebra. Note also that $Y_2^g(u, z)v = Y_2(u, z)v$ for any $u, v \in V$. Hence, (V^2, Y_2^g) is also an extension vertex operator algebra of V . By assumption, there exists a linear map $\psi : V^2 \rightarrow V^2$ such that $\psi(Y_2^g(w^1, z)w^2) = Y_2(\psi(w^1), z)\psi(w^2)$ for any $w^1, w^2 \in V^2$. This implies

$$\psi\phi(Y_1(v^1, z)v^2) = Y_2(\psi\phi(v^1), z)\psi\phi(v^2)$$

for any $v^1, v^2 \in V^1$. Hence, the vertex operator algebra V^1 is isomorphic to V^2 . The proof is complete. \square

We are now ready to show the uniqueness of holomorphic VOA of central charge 24 with Lie algebra $A_{8,3}A_{2,1}^2$. The idea is to apply Proposition 4.14. We first recall some automorphisms of $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Let φ be the diagram automorphism of sl_3 . Then we know that φ induces an automorphism of $L_{sl_3}(1,0)$, which is also denoted by φ . By a direct calculation, we can show that $L_{sl_3}(1, \dot{\Lambda}_1) \circ \varphi \cong L_{sl_3}(1, \dot{\Lambda}_2)$ and $L_{sl_3}(1, \dot{\Lambda}_2) \circ \varphi \cong L_{sl_3}(1, \dot{\Lambda}_1)$. We also need an automorphism σ of $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$, which is defined by

$$\sigma(v \otimes w) = w \otimes v,$$

for any $v, w \in L_{sl_3}(1,0)$. By a direct calculation, for any $L_{sl_3}(1,0)$ -modules M^1 and M^2 , we have $(M^1 \otimes M^2) \circ \sigma \cong M^2 \otimes M^1$.

Theorem 4.15. *Let \tilde{U}^1, \tilde{U}^2 be holomorphic vertex operator algebras such that the central charges of \tilde{U}^1, \tilde{U}^2 are equal to 24 and the Lie algebras $\tilde{U}_1^1, \tilde{U}_1^2$ are isomorphic to $A_{8,3}A_{2,1}^2$. Then there exists a vertex operator algebra isomorphism $\Phi : \tilde{U}^1 \rightarrow \tilde{U}^2$.*

Proof: We shall show that there exists a vertex operator algebra isomorphism from \tilde{U}^1 to \tilde{U}^2 . Note that \tilde{U}^1 and \tilde{U}^2 are simple current extensions of $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$, it follows from Proposition 5.3 of [20] that \tilde{U}^1 and \tilde{U}^2 viewed as extension vertex operator algebras of $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ are unique. In particular, if \tilde{U}^1 and \tilde{U}^2 viewed as $L_{sl_9}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -modules have the same decomposition, then \tilde{U}^1 is isomorphic to \tilde{U}^2 . We then assume that \tilde{U}^1, \tilde{U}^2 viewed as $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -modules have different decompositions. Let G be the subgroup of $\text{Aut}(\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0))$ generated by $\{1 \otimes \varphi \otimes 1, 1 \otimes 1 \otimes \varphi, 1 \otimes \varphi \otimes \varphi, 1 \otimes \sigma\}$. By Lemma 4.13, we can prove that there exists an automorphism $g \in G$ such that $\tilde{U}^1 \circ g$ viewed as an $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -module is isomorphic to \tilde{U}^2 . By Proposition 4.14, we know that the vertex operator algebra \tilde{U}^1 is isomorphic to \tilde{U}^2 . The proof is complete. \square

Furthermore, we have the following:

Theorem 4.16. *Let $(\tilde{U}^1, Y_1(\cdot, z)), (\tilde{U}^2, Y_2(\cdot, z))$ be holomorphic vertex operator algebras such that the central charges of \tilde{U}^1, \tilde{U}^2 are equal to 24 and the Lie algebras $\tilde{U}_1^1, \tilde{U}_1^2$ are isomorphic to $A_{8,3}A_{2,1}^2$. Assume further that \tilde{U}^1, \tilde{U}^2 viewed as $L_{sl_9}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -modules have the same decomposition, then there exists a vertex operator algebra isomorphism $\Phi : \tilde{U}^1 \rightarrow \tilde{U}^2$ such that $\Phi|_{L_{sl_9}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \text{id}$.*

Proof: First, note that both \tilde{U}^1 and \tilde{U}^2 have a vertex operator algebra isomorphic to $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. By Theorem 3.6, we know that there exists a vertex operator algebra isomorphism

$$f : \widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \rightarrow \widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$$

such that $f|_{\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \text{id}$ and that $f(Y_1(u, z)v) = Y_2(f(u), z)f(v)$ for any $u, v \in \widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Consider the linear isomorphism $\tilde{f} : \tilde{U}^1 \rightarrow \tilde{U}^2$ such that

$$\begin{aligned} \tilde{f}|_{\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} &= f, \\ \tilde{f}|_{(\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0))^\perp} &= \text{id}, \end{aligned}$$

where $(\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0))^\perp$ denotes the complement $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -module of $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ in \tilde{U}^1 . Define a new vertex operator by

$$\begin{aligned} Y_3(\cdot, z) : \tilde{U}^1 &\rightarrow \text{End}(\tilde{U}^1)[[z^{-1}, z]], \\ u &\mapsto \tilde{f}^{-1}Y_2(\tilde{f}(u), z)\tilde{f}, \end{aligned}$$

for any $u \in \tilde{U}^1$. It is easy to show that $(\tilde{U}^1, Y_3(\cdot, z))$ is a vertex operator algebra. Moreover, $(\tilde{U}^1, Y_3(\cdot, z))$ is also an extension vertex operator algebra of $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Since $(\tilde{U}^1, Y_1(\cdot, z))$ and $(\tilde{U}^1, Y_3(\cdot, z))$ are simple current extensions of $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$, we know that there exists a linear isomorphism $g : \tilde{U}^1 \rightarrow \tilde{U}^1$ such that $g|_{\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \text{id}$ and that $g(Y_1(u, z)v) = Y_3(g(u), z)g(v)$ for any $u, v \in \tilde{U}^1$ (see [20]). As a result, $\tilde{f} \circ g$ is the desired isomorphism. The proof is complete. \square

As an application of Theorem 4.16, we shall construct an automorphism of U . Note first that by Theorem 4.16 we may assume that U viewed as an $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -module is isomorphic to W^1 . Consider the automorphism $\theta \otimes \sigma$ of $\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$, we shall show that $\theta \otimes \sigma$ can be lifted to be an automorphism of U .

Theorem 4.17. *There exists an automorphism $\widetilde{\theta \otimes \sigma}$ of U such that*

$$\widetilde{\theta \otimes \sigma}(\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)) = \widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$$

and $\widetilde{\theta \otimes \sigma}|_{\widetilde{L_{sl_9}(3,0)} \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \theta \otimes \sigma$.

Proof: This follows immediately from Lemmas 3.8, 4.9, 4.10 and Theorems 3.7, 4.16. \square

5. $\widetilde{\theta \otimes \sigma}$ -TWISTED MODULE

Let U be a strongly regular holomorphic VOA of central charge 24 and $U_1 \cong A_{8,3}A_{2,1}^2$. By Theorem 4.15, the VOA structure of U is uniquely determined. Let $g = \widetilde{\theta \otimes \sigma}$ be the automorphism as given in Proposition 4.17. We shall study the unique irreducible g -twisted module U^T of U in this section. In particular, we show that the lowest (conformal) weight of U^T is 1.

Recall that U contains a full subVOA $\widetilde{L_{sl_9}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)}$ and $g|_{\widetilde{L_{sl_9}(3,0)}} = \tilde{\theta}$ and $g|_{L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \sigma$. Therefore, U^T is a direct sum of the tensor products of irreducible $\tilde{\theta}$ -twisted $\widetilde{L_{sl_9}(3,0)}$ -modules and irreducible σ -twisted $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -modules. Since $\widetilde{L_{sl_9}(3,0)}$ is an extension of $L_{sl_9}(3,0)$, irreducible $\tilde{\theta}$ -twisted modules of $\widetilde{L_{sl_9}(3,0)}$ are direct sum of irreducible θ -twisted $L_{sl_9}(3,0)$ -modules.

5.1. \mathbb{Z}_2 -twisted module of lattice VOA. First, let us recall the construction of irreducible \mathbb{Z}_2 -twisted modules of lattice VOA from [14] (see also [26]).

Let L be an even lattice with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ and let σ be an isometry of L of order 2. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and extend the bilinear form $\langle \cdot, \cdot \rangle$ \mathbb{C} -bilinearly to \mathfrak{h} . Set $\mathfrak{h}_{(0)} = \{\mathfrak{h} \mid \sigma x = x\}$ and $\mathfrak{h}_{(1)} = \{\mathfrak{h} \mid \sigma x = -x\}$. For $i = 0, 1$, let P_i be the natural projection of \mathfrak{h} to $\mathfrak{h}_{(i)}$. We also use $x_{(i)}$ to denote $P_i(x)$ for any $x \in \mathfrak{h}$.

The twisted affine algebra $\widehat{\mathfrak{h}}[\sigma]$ is the Lie algebra

$$\widehat{\mathfrak{h}}[\sigma] = \mathfrak{h}_{(0)} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{h}_{(1)} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

with the bracket given by

$$[x \otimes t^m, y \otimes t^n] = m\langle x, y \rangle \delta_{m+n,0}c \quad \text{and} \quad [c, \widehat{\mathfrak{h}}_{\mathbb{Z}+\frac{1}{2}}] = 0$$

where $x, y \in \mathfrak{h}_{(i)}$ and $m, n \in \mathbb{Z} + \frac{i}{2}, i = 0, 1$. The Lie algebra $\widehat{\mathfrak{h}}[\sigma]$ has a triangular decomposition given by

$$\widehat{\mathfrak{h}}[\sigma] = \widehat{\mathfrak{h}}^+[\sigma] \oplus \widehat{\mathfrak{h}}^0[\sigma] \oplus \widehat{\mathfrak{h}}^-[\sigma]$$

where $\widehat{\mathfrak{h}}^{\pm}[\sigma] = \bigoplus_{n=1}^{\infty} (\mathfrak{h}_{(0)} \otimes t^{\pm n} \oplus \mathfrak{h}_{(1)} \otimes t^{\pm(n-\frac{1}{2})})$ and $\widehat{\mathfrak{h}}^0[\sigma] = \mathfrak{h}_{(0)} \oplus \mathbb{C}c$.

Let $N = (1 - P_0)\mathfrak{h} \cap L = \{\alpha \in L \mid \langle \alpha, \mathfrak{h}_{(0)} \rangle = 0\}$ and $M = (1 - \sigma)L$. Note that $M < N$ since $\langle M, \mathfrak{h}_{(0)} \rangle = 0$. Let $\langle \kappa \rangle$ be a cyclic group of order 2. On N , we define

$$C_N(\alpha, \beta) = \kappa^{\langle \alpha, \beta \rangle}, \quad \text{and} \quad R = \{\alpha \in N \mid C_N(\alpha, N) = 1\}.$$

Let

$$1 \longrightarrow \langle \kappa \rangle \longrightarrow \widehat{N} \xrightarrow{\varphi} N \longrightarrow 1$$

be the central extension of N associated with the commutator map C_N and let $\widehat{\sigma}$ be a lift of σ in $\text{Aut}(\widehat{N})$, i.e., $\varphi(\widehat{\sigma}(a)) = \sigma(\varphi(a))$ for any $a \in \widehat{N}$. Set $K = \{a\widehat{\sigma}(a)^{-1} \mid a \in \widehat{N}\}$. Then K is an index 2 subgroup of \widehat{M} .

Proposition 5.1 ([43]). *For any irreducible character $\chi : \widehat{R}/K \rightarrow \mathbb{C}$ with $\chi(\kappa K) = -1$, there is a unique irreducible \widehat{N}/K -module T_χ such that \widehat{R} acts according to χ . Moreover, $\dim T_\chi = |N/R|^{1/2}$.*

Let $L^* = \{\alpha \in L \otimes_{\mathbb{Z}} \mathbb{C} \mid \langle \alpha, L \rangle \subset \mathbb{Z}\}$ be the dual lattice of L . For any coset $\lambda + P_0(L) \in P_0(L^*)/P_0(L)$ and an irreducible character $\chi \in \text{Irr}(\widehat{R}/K)$, denote

$$V_L^{T_{\chi,\lambda}} = S(\widehat{\mathfrak{h}}^-[\sigma]) \otimes \mathbb{C}[\lambda + P_0(L)] \otimes T_\chi.$$

It is shown in [14] (see also [43]) that $V_L^{T_{\chi,\lambda}}$ is an irreducible $\widehat{\sigma}$ -twisted module of V_L . Moreover, the lowest (conformal) weight of $V_L^{T_{\chi,\lambda}}$ is given by

$$\frac{\text{rank}(N)}{16} + \frac{n_\lambda}{2},$$

where $n_\lambda = \min\{\langle \alpha, \alpha \rangle \mid \alpha \in \lambda + P_0(L)\}$.

5.2. σ -twisted module of $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Next we study the σ -twisted modules for $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$. Recall that $L_{sl_3}(1,0)$ is isomorphic to the lattice VOA V_{A_2} and hence, $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0) \cong V_{A_2 \oplus A_2}$.

Let $\sigma : A_2 \oplus A_2 \rightarrow A_2 \oplus A_2$ be an isometry defined by $\sigma(\alpha, \beta) = (\beta, \alpha)$ and set $A^+ = \{(\alpha, \alpha) \mid \alpha \in A_2\}$ and $A^- = \{(\alpha, -\alpha) \mid \alpha \in A_2\}$. Then A^+ and A^- are the eigenlattices of σ of eigenvalues $+1$ and -1 , respectively.

Let $P_0 = \frac{1}{2}(1 + \sigma)$ be the natural projection from $(A_2 \oplus A_2)^*$ to $\mathfrak{h}_{(0)}$. Then

$$P_0(A_2 \oplus A_2) = \frac{1}{2}A^+ \quad \text{and} \quad (1 - P_0)\mathfrak{h} \cap (A_2^2) = A^- = (1 - \sigma)(A_2^2).$$

Moreover, we have $P_0((A_2^*)^2) = \frac{1}{2}\{(\alpha, \alpha) \mid \alpha \in A_2^*\}$ and $|P_0((A_2^*)^2)/P_0(A_2^2)| = 3$.

Let $P_0((A_2^*)^2) = P_0(A_2^2) \cup (\lambda_1 + P_0(A_2^2)) \cup (\lambda_2 + P_0(A_2^2))$ be the coset decomposition of $P_0((A_2^*)^2)$ in $P_0((A_2^*)^2)$. Then, by [14] (see also [5]), we have the following lemma.

Lemma 5.2. *There are 3 inequivalent irreducible σ -twisted modules for $V_{A_2 \oplus A_2}$ and they are given by*

$$\begin{aligned} W_0 &= S(\widehat{\mathfrak{h}}^-[\sigma]) \otimes \mathbb{C}[P_0(A_2^2)] \otimes T, \\ W_1 &= S(\widehat{\mathfrak{h}}^-[\sigma]) \otimes \mathbb{C}[\lambda_1 + P_0(A_2^2)] \otimes T, \text{ and} \\ W_2 &= S(\widehat{\mathfrak{h}}^-[\sigma]) \otimes \mathbb{C}[\lambda_2 + P_0(A_2^2)] \otimes T, \end{aligned}$$

where T is a one-dimensional irreducible module of \widehat{A}^- . The lowest weight of W_0 is $2/16 = 1/8$ and the lowest weights of W_1 and W_2 are $1/8 + 1/6 = 7/24$.

5.3. Irreducible θ -twisted modules of $L_{sl_9}(3, 0)$. Next we consider the irreducible θ -twisted modules of $L_{sl_9}(3, 0)$.

Lemma 5.3. *There are exactly 5 irreducible θ -invariant modules for $L_{sl_9}(3, 0)$, namely, $L_{sl_9}(3, 0)$, $L_{sl_9}(3, \ddot{\Lambda}_1 + \ddot{\Lambda}_8)$, $L_{sl_9}(3, \ddot{\Lambda}_2 + \ddot{\Lambda}_7)$, $L_{sl_9}(3, \ddot{\Lambda}_3 + \ddot{\Lambda}_6)$, and $L_{sl_9}(3, \ddot{\Lambda}_4 + \ddot{\Lambda}_5)$.*

Proof. The lemma follows immediately from Lemma 3.8. \square

By [17, Theorem 1.1], we also have the following result.

Corollary 5.4. *There are exactly 5 inequivalent irreducible θ -twisted modules for $L_{sl_9}(3, 0)$.*

Now let $L = A_8^3$ be a root lattice of type A_8^3 . For explicit calculations, we use the standard model for A_8 , i.e.,

$$A_8 = \{(a_1, \dots, a_9) \in \mathbb{Z}^9 \mid \sum_{i=1}^9 a_i = 0\}.$$

The following lemma can be obtained by a direct calculation.

Lemma 5.5. *Every coset of $A_8/2A_8$ contains a vector of norm ≤ 8 and the coset representatives (with minimal norm) are given as follows:*

Norm	Representatives	# of cosets
0	0	1
2	$\pm(1, -1, 0^7)$	36
4	$\pm(1^2, -1^2, 0^5)$	126
6	$\pm(1^3, -1^3, 0^3)$	84
8	$\pm(1^4, -1^4, 0)$	8

Let X and Y be sublattices of A_8 such that $X/2A_8$ and $Y/2A_8$ are maximal totally singular subspaces of $A_8/2A_8$ and $X+Y = A_8$. Recall that $A_8/2A_8$ forms a non-singular quadratic space associated with the standard quadratic form $q(\alpha + 2A_8) = \langle \alpha, \alpha \rangle / 2 \pmod{2}$ since $\det(A_8) = 9$.

Set $\mu_{i,j} = \eta_i - \eta_j$ for $i \neq j$ and $\eta = \eta_1 + \eta_2 + \eta_3$ and let Φ be the sublattice of $L = A_8^3$ spanned by

$$\mu_{1,2}(A_8) \cup \eta(X) \cup 2L$$

and let Ψ be the sublattice spanned by

$$\mu_{2,3}(A_8) \cup \eta(Y) \cup 2L.$$

Then $\Phi/2L$ and $\Psi/2L$ are maximal totally singular subspaces of $L/2L$ and $\Phi + \Psi = L$. Note that $\mu_{i,j}(A_8) \perp \eta(A_8)$.

Let χ_0 be an irreducible character of $\widehat{\Phi}/K$ such that $\chi_0(\iota(e_\alpha)) = 1$ and $\chi_0(\kappa K) = -1$. Then

$$T = \text{Ind}_{\widehat{\Phi}/K}^{\widehat{L}/K} F_{\chi_0},$$

where F_{χ_0} is the irreducible $\widehat{\Phi}/K$ -module affording the character χ_0 .

Take $0 \neq t_0 \in F_{\chi_0}$. Then $F_{\chi_0} = \mathbb{C}t_0$ and

$$T = \text{Span}_{\mathbb{C}} \{e_{\mu_{2,3}(\beta)+\eta(\gamma)} \cdot t_0 \mid \beta \in A_8, \gamma \in Y\}$$

For simplicity, we set $t_{(\beta,\gamma)} = e_{\mu_{2,3}(\beta)+\eta(\gamma)} \cdot t_0$.

Recall from Sec. 3.4 that the lattice VOA $V_{A_8^3}$ contains a subVOA isomorphic to the affine VOA $L_{sl_9}(3, 0)$, which is generated by

$$\tilde{h} = \eta(h)(-1) \cdot \mathbf{1} \quad \text{for } h \in A_8 \otimes_{\mathbb{Z}} \mathbb{C},$$

$$E_\alpha = \iota(e_{\eta_1(\alpha)}) + \iota(e_{\eta_2(\alpha)}) + \iota(e_{\eta_3(\alpha)}) \quad \text{for } \alpha \in (A_8)_2.$$

The conformal element Ω of $L_{sl_9}(3, 0)$ is given by

$$\Omega = \omega_E + \frac{3}{4}\omega_P - \frac{1}{12} \sum_{\substack{\alpha \in (A_8)_2 \\ 1 \leq i < j \leq 3}} e_{\mu_{i,j}(\alpha)},$$

where $E = \eta(A_8)$, $P = \{(\alpha, \beta, \gamma) \in A_8^3 \mid \alpha + \beta + \gamma = 0\}$ and ω_M denotes the conformal element of the lattice VOA V_M .

Next we shall construct some explicit eigenvectors of Ω_1 on T .

Lemma 5.6. *For $\alpha \in (A_8)_2$ and $(\beta, \gamma) \in A_8 \times Y$, we have*

$$\begin{aligned} & (e_{\mu_{1,2}(\alpha)} + e_{\mu_{2,3}(\alpha)} + e_{\mu_{1,3}(\alpha)})_1 t_{(\beta,\gamma)} \\ &= \begin{cases} \frac{1}{16} t_{(\beta,\gamma)} & \text{if } \langle \alpha, \beta \rangle = 0 \pmod{2}, \\ \frac{1}{16} (2t_{(\alpha+\beta,\gamma)} - t_{(\beta,\gamma)}) & \text{if } \langle \alpha, \beta \rangle = 1 \pmod{2}. \end{cases} \end{aligned}$$

Proof. Recall from Sec. 3.3 that

$$e_{\mu_{1,2}(\alpha)} e_{\mu_{2,3}(\beta)} = (-1)^{\langle \alpha, \beta \rangle} e_{\mu_{2,3}(\beta)} e_{\mu_{1,2}(\alpha)};$$

$$e_{\mu_{2,3}(\alpha)} e_{\mu_{2,3}(\beta)} = e_{\mu_{2,3}(\alpha+\beta)};$$

$$e_{\mu_{1,3}(\alpha)} = (-1)^{\epsilon_0(\alpha, \alpha)} e_{\mu_{2,3}(\alpha)} e_{\mu_{1,2}(\alpha)} = -e_{\mu_{2,3}(\alpha)} e_{\mu_{1,2}(\alpha)}.$$

Since $e_{\mu_{1,2}(\alpha)} \cdot t_0 = t_0$ for all $\alpha \in (A_8)_2$, we have

$$\begin{aligned} & (e_{\mu_{1,2}(\alpha)} + e_{\mu_{2,3}(\alpha)} + e_{\mu_{1,3}(\alpha)})_1 t_{(\beta,\gamma)} \\ &= \frac{1}{24} (e_{\mu_{1,2}(\alpha)} + e_{\mu_{2,3}(\alpha)} + e_{\mu_{1,3}(\alpha)}) \cdot e_{\mu_{2,3}(\beta)} e_{\eta(\gamma)} \cdot t_0 \\ &= \frac{1}{16} ((-1)^{\langle \alpha, \beta \rangle} t_{(\beta,\gamma)} + t_{(\alpha+\beta,\gamma)} - (-1)^{\langle \alpha, \beta \rangle} t_{(\alpha+\beta,\gamma)}). \end{aligned}$$

Thus we have the desired conclusion. \square

Notation 5.7. For $n = 0, 1, 2, 3, 4$, let \mathcal{C}_{2n} be the set of cosets of $2A_8$ in A_8 with minimal norm $2n$, i.e., $\mathcal{C}_{2n} = \{\alpha + 2A_8 \mid \min\{\langle a, a \rangle \mid a \in \alpha + 2A_8\} = 2n\}$. Note that a coset $\alpha + 2A_8 \in A_8/2A_8$ is in \mathcal{C}_{2n} if it contains a vector of the shape $(1^n, -1^n, 0^{9-2n})$.

Lemma 5.8. Let $\mu = (1, -1, 0^7)$ and denote $v_0 = t_0$ and

$$v_1 = 21t_{(\mu,0)} + \sum_{\substack{\langle \beta, \mu \rangle = 0 \pmod{2} \\ \beta + 2A_8 \in \mathcal{C}_2 \setminus \{\mu + 2A_8\}}} t_{(\beta,0)} - 3 \sum_{\substack{\langle \beta, \mu \rangle = 1 \pmod{2} \\ \beta + 2A_8 \in \mathcal{C}_2}} t_{(\beta,0)}.$$

Then we have

$$\Omega_1 v_0 = \frac{7}{8} v_0, \quad \text{and} \quad \Omega_1 v_1 = \frac{29}{24} v_1.$$

Proof. First we note that

$$(\omega_E)_1 t = \frac{8}{16} t \quad \text{and} \quad (\omega_P)_1 t = \frac{16}{16} t$$

for any $t \in T$. Note that $\text{rank}(E) = 8$ and $\text{rank}(P) = 16$.

For $v_0 = t_0$, we have

$$\left(\sum_{\substack{\alpha \in (A_8)_2 \\ 1 \leq i < j \leq 3}} e_{\mu_{i,j}(\alpha)} \right)_1 v_0 = \frac{72}{16} v_0$$

by Lemma 5.6. Notice that there are 72 elements in $(A_8)_2$. Hence,

$$\Omega_1 v_0 = \frac{1}{2} v_0 + \frac{3}{4} v_0 - \frac{1}{12} \cdot \frac{72}{16} v_0 = \frac{7}{8} v_0.$$

For any $\beta + 2A_8 \in \mathcal{C}_2$, we have

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 0 \pmod{2}\}| = 44;$$

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1 \pmod{2}\}| = 28.$$

Let $\mu = (1, -1, 0^7)$. If $\beta + 2A_8 = \mu + 2A_8$, then $\langle \alpha, \beta \rangle = 1 \pmod{2}$ implies

$$\langle \alpha + \beta, \mu \rangle = \langle \alpha + \beta, \beta \rangle = 1 \pmod{2}$$

and there are 28 such $\alpha \in (A_8)_2$.

If $\beta + 2A_8 \neq \mu + 2A_8$ and $\langle \beta, \mu \rangle = 0 \pmod{2}$, then

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 0 \pmod{2}\}| = 20;$$

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 1 \pmod{2}\}| = 8.$$

If $\langle \beta, \mu \rangle = 1 \pmod{2}$, then

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 0 \pmod{2}, \alpha + \beta + 2A_8 \neq \mu + 2A_8\}| = 12;$$

$$|\{\alpha \in (A_8)_2 \mid \alpha + \beta + 2A_8 = \mu + 2A_8\}| = 2;$$

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 1 \pmod{2}\}| = 14.$$

Then by Lemma 5.6, we have

$$\begin{aligned}
& 16 \left(\sum_{\substack{\alpha \in (A_8)_2 \\ 1 \leq i < j \leq 3}} e_{\mu_{i,j}(\alpha)} \right)_1 v_1 \\
&= (21 \cdot (44 - 28) - 3 \cdot 2 \cdot 28) t_{(\mu,0)} \\
&\quad + ((44 - 28) + 2 \cdot 20 - 3 \cdot 2 \cdot 8) \sum_{\substack{\langle \beta, \mu \rangle = 0 \pmod{2} \\ \beta + 2A_8 \in \mathcal{C}_{2n} \setminus \{\mu + 2A_8\}}} t_{(\beta,0)} \\
&\quad - (3 \cdot (44 - 28) + 3 \cdot 2 \cdot 14 - 2 \cdot 12 - 21 \cdot 2 \cdot 2) \sum_{\substack{\langle \beta, \mu \rangle = 1 \pmod{2} \\ \beta + 2A_8 \in \mathcal{C}_{2n}}} t_{(\beta,0)} \\
&= 8 \left(21 t_{(\mu,0)} + \sum_{\substack{\langle \beta, \mu \rangle = 0 \pmod{2} \\ \beta + 2A_8 \in \mathcal{C}_{2n} \setminus \{\mu + 2A_8\}}} t_{(\beta,0)} - 3 \sum_{\substack{\langle \beta, \mu \rangle = 1 \pmod{2} \\ \beta + 2A_8 \in \mathcal{C}_{2n}}} t_{(\beta,0)} \right) = 8 v_1.
\end{aligned}$$

Hence we have

$$\Omega_1 v_1 = \frac{1}{2} v_1 + \frac{3}{4} v_1 - \frac{1}{12} \cdot \frac{8}{16} v_1 = \frac{29}{24} v_1.$$

□

Notation 5.9. For any $\mu \in A_8$, we denote

$$P_{2n}^\mu = \sum_{\substack{\langle \beta, \mu \rangle = 0 \pmod{2} \\ \beta + 2A_8 \in \mathcal{C}_{2n}}} t_{(\beta,0)}, \quad \text{and} \quad N_{2n}^\mu = \sum_{\substack{\langle \beta, \mu \rangle = 1 \pmod{2} \\ \beta + 2A_8 \in \mathcal{C}_{2n}}} t_{(\beta,0)}.$$

Lemma 5.10. Let $\mu_4 = (1^2, -1^2, 0^5)$, $\mu_6 = (1^3, -1^3, 0^3)$ and $\mu_8 = (1^4, -1^4, 0)$ and denote $v_2 = 10P_2^{\mu_4} - 7N_2^{\mu_4}$, $v_3 = 6P_2^{\mu_6} - 7N_2^{\mu_6}$, and $v_4 = 5P_8^{\mu_8} - N_8^{\mu_8}$. Then we have

$$\Omega_1 v_2 = \frac{59}{48} v_2, \quad \Omega_1 v_3 = \frac{55}{48} v_3, \quad \text{and} \quad \Omega_1 v_4 = \frac{17}{16} v_4.$$

Proof. The proof is similar to the previous lemma. First we have

$$(\omega_E + \frac{3}{4} \omega_P)_1 t = \frac{5}{4} t$$

for any $t \in T$.

Let $\mu = \mu_4 = (1^2, -1^2, 0^5)$. For any $\beta + 2A_8 \in \mathcal{C}_2$, we have

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 0 \pmod{2}\}| = 44;$$

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1 \pmod{2}\}| = 28.$$

If $\langle \beta, \mu \rangle = 0 \pmod{2}$, then

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 0 \pmod{2}\}| = 8;$$

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 1 \pmod{2}\}| = 20.$$

If $\langle \beta, \mu \rangle = 1 \pmod{2}$, then

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 0 \pmod{2}\}| = 14;$$

$$|\{\alpha \in (A_8)_2 \mid \langle \beta, \alpha \rangle = 1, \langle \alpha + \beta, \mu \rangle = 1 \pmod{2}\}| = 14.$$

Then by Lemma 5.6, we have

$$\begin{aligned} & 16 \left(\sum_{\substack{\alpha \in (A_8)_2 \\ 1 \leq i < j \leq 3}} e_{\mu_{i,j}(\alpha)} \right)_1 v_2 \\ &= (10 \cdot (16 + 16) - 40 \cdot 7) P_2^{\mu_4} - (7 \cdot (16 + 28) - 28 \cdot 10) N_2^{\mu_4} \\ &= 40 P_2^{\mu_4} - 28 N_2^{\mu_4} = 4 v_2. \end{aligned}$$

Hence we have

$$\Omega_1 v_2 = \frac{5}{4} v_1 - \frac{1}{12} \cdot \frac{4}{16} v_2 = \frac{59}{48} v_2.$$

The other cases can be proved by the similar method. \square

By Corollary 5.4 and the lemmas above, we have the following result.

Lemma 5.11. *There are 5 inequivalent irreducible θ -twisted modules for $L_{sl_9}(3, 0)$ and their lowest conformal weights are $7/8$, $29/24$, $59/48$, $55/48$ and $17/16$.*

Lemma 5.12. *Let α be a root of A_8 and $(\beta, \gamma) \in A_8 \times Y$. Then*

$$E_\alpha \cdot t_{(\beta, \gamma)} = \begin{cases} t_{(\beta, \gamma + \alpha)} & \text{if } \langle \alpha, \beta \rangle = 0 \pmod{2}, \\ 2t_{(\beta + \alpha, \gamma + \alpha)} - t_{(\beta, \gamma + \alpha)} & \text{if } \langle \alpha, \beta \rangle = 1 \pmod{2}. \end{cases}$$

Proof. First we note that

$$e_{\eta_1(\alpha)} = e_{\eta(\alpha)} e_{\mu_{2,3}(\alpha)} e_{-2\eta_2(\alpha)},$$

$$e_{\eta_2(\alpha)} = -e_{\eta(\alpha)} e_{\mu_{2,3}(\alpha)} e_{-\mu_{1,2}(\alpha)} e_{-2\eta_3(\alpha)},$$

$$e_{\eta_3(\alpha)} = e_{\eta(\alpha)} e_{\mu_{1,2}(\alpha)} e_{-2\eta_2(\alpha)}.$$

Thus,

$$\begin{aligned} E_\alpha &= (e_{\eta_1(\alpha)} + e_{\eta_2(\alpha)} + e_{\eta_3(\alpha)}) \cdot t_{\beta, \gamma} \\ &= t_{(\beta + \alpha, \gamma + \alpha)} - (-1)^{\langle \alpha, \beta \rangle} t_{(\beta + \alpha, \gamma + \alpha)} + (-1)^{\langle \alpha, \beta \rangle} t_{(\beta, \gamma + \alpha)} \end{aligned}$$

and we have the desired result. \square

Lemma 5.13. *Let M be the θ -twisted $L_{sl_9}(3, 0)$ -module generated by t_0 . Let $M(0)$ be the top module of an irreducible (twisted or untwisted) module M . Then we have*

$$M(0) = \text{Span}_{\mathbb{C}} \{t_{(0, \gamma)} \mid \gamma \in Y\}.$$

In particular, $M(0)$ has dimension 16.

Proposition 5.14. *Let U be a holomorphic VOA of central charge 24 and $U_1 \cong A_{8,3}A_{2,1}^2$. Let $g = \widetilde{\theta \otimes \sigma}$ be an involution of U as given in Theorem 4.17. Let U^T be the unique irreducible g -twisted module of U . Then the lowest conformal weight of U^T is 1 and $\dim(U_1^T) \geq 16$.*

Proof. We first note that U^T is a direct sum of the tensor products of irreducible $\tilde{\theta}$ -twisted $L_{sl_9}(3,0)$ -modules and irreducible σ -twisted $L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -modules. Moreover, every $\tilde{\theta}$ -twisted $L_{sl_9}(3,0)$ -module is a direct sum of irreducible θ -twisted $L_{sl_9}(3,0)$ -modules.

By [17, Theorem 1.6], the conformal weights of U^T are in $\frac{1}{4}\mathbb{Z}$. Then by Lemmas 5.2 and 5.11, the lowest conformal weights for irreducible $\widetilde{\theta \otimes \sigma}$ -twisted $L_{sl_9}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)$ -submodules of U^T are $1 (= 7/8 + 1/8)$ and $3/2 (= 29/24 + 7/24)$. Therefore, the lowest conformal weight of U^T is 1. That $\dim(U_1^T) \geq 16$ follows from Lemma 5.13. \square

6. $\widetilde{\theta \otimes \sigma}$ -ORBIFOLD CONSTRUCTION

Let U , U^T and g be defined as in Proposition 5.14. Since the conformal weights of U^T are in $\frac{1}{2}\mathbb{Z}$, we can apply an orbifold construction to U using g (see [24, Theorem 5.15] and [8]) and obtain a strongly regular holomorphic VOA

$$\tilde{U}(g) = U^g \oplus (U^T)_{\mathbb{Z}}$$

of central charge 24. The following lemma is a generalization of [41, Theorem 4.3] (see also [51]).

Lemma 6.1. *Let U and g be as above. Then*

$$\dim U_1 + \dim \tilde{U}(g)_1 = 3 \dim(U^g)_1 + 24(1 - \dim(U^T)_{1/2}).$$

Proof. Let $Z_U(g, \tau) = q^{-c/24} \sum_{n=0}^{\infty} \text{Tr } g|_{U_n} q^n$ be the trace function of g on U and let $Z_{U^T}(\tau) = q^{-c/24} \sum_{n=1}^{\infty} \dim(U^T)_n q^{n/2}$ be the character of U^T , where $q = e^{2\pi\sqrt{-1}\tau}$ and τ is in the upper half plane \mathfrak{H} .

It was proved in [17] that $Z_U(g, \tau)$ and $Z_{U^T}(\tau)$ both converge to holomorphic functions in \mathfrak{H} and

$$Z_U(g, S\tau) = Z_U(g, -\frac{1}{\tau}) = \lambda Z_{U^T}(\tau)$$

for some $\lambda \in \mathbb{C}$. Moreover, it was proved in [24, Proposition 5.5] that $\lambda = 1$. Therefore, U and U^T satisfy Assumptions (A1) and (A2) of [41, Section 4.2]. Hence the proof of Theorem 4.3 of [41] still holds for U and g and we have

$$\dim U_1 + \dim \tilde{U}(g)_1 = 3 \dim(U^g)_1 + 24(1 - \dim(U^T)_{1/2}),$$

as desired. \square

By a direct calculation, we also have the following.

Lemma 6.2. *Let U , U^T and g be defined as above.*

1. *The weight one Lie algebra of U^g has the type B_4A_2 and has dimension 44.*
2. $\dim(\tilde{U}(g)_1) = 60$.

Proof. Recall that $g|_{L_{sl_9}(3,0) \otimes L_{sl_3}(1,0) \otimes L_{sl_3}(1,0)} = \theta \otimes \sigma$. Since the fixed point Lie algebra of θ on sl_9 has type B_4 and the fixed point of σ on $sl_3 \oplus sl_3$ has type A_2 , we have (1).

For (2), we have $\dim(\tilde{U}(g)_1) = 3 \times 44 + 24 - 96 = 60$ by Lemma 6.1. \square

Theorem 6.3. *Let U , U^T and g be defined as above. Then $\tilde{U}(g) = U^g \oplus (U^T)_{\mathbb{Z}}$ is a strongly regular holomorphic VOA of central charge 24 and $\tilde{U}(g)_1$ has the type $F_{4,6}A_{2,2}$.*

Proof. Since $\dim(\tilde{U}(g)_1) = 60$, the ratio $\frac{h^\vee}{k} = \frac{60-24}{24} = 3/2$ (cf. [20]). Therefore, the dual Coxeter number of any simple ideal of $\tilde{U}(g)_1$ must be divisible by 3 and hence a simple ideal must have the type A_2 , C_2 , A_5 , C_5 , D_4 , or F_4 . That $\dim(\tilde{U}(g)_1) = 60$ implies that $\tilde{U}(g)_1$ has the type $C_{2,2}^6$, $D_{4,4}A_{2,2}^4$ or $F_{4,6}A_{2,2}$. Since $\tilde{U}(g)_1$ contains a Lie subalgebra of type B_4A_2 , $\tilde{U}(g)_1 = F_{4,6}A_{2,2}$ is the only possibility. \square

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